Improved Bounds for the Flat Wall Theorem*

Julia Chuzhoy†

Abstract

The Flat Wall Theorem of Robertson and Seymour states that there is some function f, such that for all integers $w, t > 1$, every graph G containing a wall of size $f(w,t)$, must contain either (i) a K_t-minor; or (ii) a small subset $A \subset V(G)$ of vertices, and a flat wall of size w in $G \setminus A$. Kawarabayashi, Thomas and Wollan recently showed a self-contained proof of this theorem with the following two sets of parameters: (1) $f(w,t) = \Theta(t^{24}(t^2+w))$ with $|A| = O(t^{24})$, and (2) $f(w,t) = w^{2^{\Theta(t^{24})}}$ with $|A| \le t-5$. The latter result gives the best possible bound on $|A|$. In this paper we improve their bounds to $f(w,t) = \Theta(t(t+w))$ with $|A| \le t-5$. For the special case where the maximum vertex degree in G is bounded by D, we show that, if G contains a wall of size $\Omega(Dt(t+w))$, then either G contains a K_t-minor, or there is a flat wall of size w in G. This setting naturally arises in algorithms for the Edge-Disjoint Paths problem, with $D \le 4$. Like the proof of Kawarabayashi et al., our proof is self-contained, except for using a well-known theorem on routing pairs of disjoint paths. We also provide efficient algorithms that return either a model of the K_t-minor, or a vertex set A and a flat wall of size w in $G \setminus A$.

We complement our result for the low-degree scenario by proving an almost matching lower bound: namely, for all integers $w, t > 1$, there is a graph G, containing a wall of size $\Omega(wt)$, such that the maximum vertex degree in G is 5, and G contains no flat wall of size w, and no K_t-minor.

1 Introduction

The main combinatorial object studied in this paper is a wall. In order to define a wall W of height h and width r, or an $(h \times r)$-wall, we start from a grid of height h and width $2r$. Let $C_1, \ldots, C_{2r}$ be the columns of the grid in their natural left-to-right order. For each column C_j, let $e_1^j, e_2^j, \ldots, e_{h-1}^j$ be the edges of C_j, in their natural top-to-bottom order. If j is odd, then we delete all edges e_i^j where i is even. If j is even, then we delete all edges e_i^j where i is odd. We then remove all vertices of the resulting graph whose degree is 1. This final graph, denoted by $\hat{W}$, is called an *elementary* $(h \times r)$-*wall* (see Figure 1). The *pegs* of $\hat{W}$ are all the vertices on its outer boundary that have degree 2. An $(h \times r)$-wall W is simply a subdivision of the elementary $(h \times r)$-wall $\hat{W}$, and the pegs of W are defined to be the vertices of W that serve as the pegs of $\hat{W}$. We sometimes refer to a $(w \times w)$-wall as a *wall of size* w.

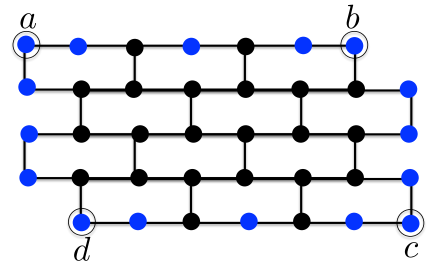

Figure 1: An elementary wall of height 5 and width 4, with the corners circled.

The well-known Excluded Grid Theorem of Robertson and Seymour [RS86] states that there is some function $g : \mathbb{Z}^+ \to \mathbb{Z}^+$, such that for every integer $w \ge 1$, every graph of treewidth at least $g(w)$ contains the $(w \times w)$-grid as a minor. Equivalently, if the treewidth of G is $g(w)$, then G contains a wall of size $\Omega(w)$. This important theorem has found many applications in graph theory and algorithms. However, in some scenarios it is useful to have more structure than that provided by the presence of a large wall in a graph. The Flat Wall Theorem helps provide this additional structure, and it is used, for example, in algorithms for the Node-Disjoint Paths problem [RS95]. We start with some basic definitions and results that are needed in order to state the Flat Wall Theorem.

Suppose we are given a graph $G = (V, E)$ with four special vertices s_1, t_1, s_2, t_2. In the Two-Disjoint-Paths problem, our goal is to find two disjoint paths P_1 and P_2 in G, with P_1 connecting s_1 to t_1, and P_2 connecting s_2 to t_2. The well-known Two-Disjoint-Paths Theorem [Jun70, RS90, Sey06, Shi80, Tho80] states that either there is a solution to the Two-Disjoint-Paths problem, or G can "almost" be drawn inside a disc in the plane, with s_1, s_2, t_1, t_2 appearing on its boundary in this circular order. In order to define the specific notion of the "almost" drawing, we first need to define C-reductions.

Recall that a *separation* in a graph G is a pair X, Y of sub-graphs of G, such that $G = X \cup Y$, and

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†Toyota Technological Institute, Chicago, IL 60637. Email: cjulia@ttic.edu. Supported in part by NSF grant CCF-1318242.

$E(X) \cap E(Y) = \emptyset$. The *order* of the separation is $|V(X) \cap V(Y)|$. Let C be any set of vertices in graph G, and let (X, Y) be any separation of H of order at most 3 with $C \subseteq V(Y)$. Assume further that all vertices of $X \cap Y$ are connected inside graph X. Let $\tilde{G}$ be the graph obtained from Y by adding the edges connecting all pairs of vertices in $X \cap Y$. Then we say that $\tilde{G}$ is an *elementary C-reduction* of H. Observe that if $s_1, t_1, s_2, t_2 \in C$, then there is a solution to the Two-Disjoint-Paths problem in G iff there is such a solution in $\tilde{G}$. This is since at most one of the two paths P_1, P_2 may contain the vertices of $X \setminus Y$. We say that a graph G^* is a *C-reduction* of G iff it can be obtained from G by a sequence of elementary C-reductions. The Two-Disjoint-Paths Theorem states that either there is a feasible solution to the Two-Disjoint-Paths problem in G, or some C-reduction of G, for $C = \{s_1, s_2, t_1, t_2\}$, can be drawn inside a disc in the plane, with the vertices of C appearing on the boundary of the disc, in the circular order (s_1, s_2, t_1, t_2).

More generally, let C be any set of vertices of G, and let $\tilde{C}$ be any circular ordering of the vertices of C. A $\tilde{C}$-cross in G is a pair P_1, P_2 of disjoint paths, whose endpoints are denoted by s_1, t_1 and s_2, t_2, respectively, such that $s_1, s_2, t_1, t_2 \in C$, and they appear in $\tilde{C}$ in this circular order. A more general version of the Two-Disjoint-Paths Theorem [Jun70, RS90, Sey06, Shi80, Tho80] states that either G contains a $\tilde{C}$-cross, or some C-reduction of G can be drawn inside a disc in the plane, with the vertices of C appearing on the boundary of the disc, in the order specified by $\tilde{C}$. In the latter case, we say that graph G is *$\tilde{C}$-flat*.

Given a wall W, let $\Gamma(W)$ be the outer boundary of W, and let C be the set of the pegs of W. We say that W is a *flat wall* in G iff there is a separation (X, Y) of G, with $W \subseteq Y$, $X \cap Y$ contained in $\Gamma(W)$, and the set C of all pegs of W contained in $X \cap Y$, such that, if we denote $Z = X \cap Y$, and $\tilde{Z}$ is the ordering of the vertices of Z induced by $\Gamma(W)$, then graph Y is $\tilde{Z}$-flat.

We are now ready to state the Flat Wall Theorem (a more formal statement appears in Section 2). There are two functions f and g, such that, for any integers $w, t > 1$, for any graph G containing a wall of size $f(w, t)$, either (i) G contains a K_t-minor, or (ii) there is a set A of at most $g(t)$ vertices in G, and a flat wall of size w in $G \setminus A$. A somewhat stronger version of this theorem was originally proved by Robertson and Seymour [RS95], with $g(t) = O(t^2)$; however, they do not provide explicit bounds on $f(w, t)$. Giannopoulou and Thilikos [GT13] showed a proof of this theorem with $g(t) = t - 5$, obtaining the best possible bound on $|A|$, but they also do not provide explicit bounds on $f(w, t)$. Recently, Kawarabayashi, Thomas and Wollan [KTW12] gave a self-contained proof of the theorem in the following two settings: with $g(t) = O(t^{24})$ they achieve $f(w, t) = \Theta(t^{24}(t^2 + w))$, and with $g(t) = t - 5$, they obtain $f(w, t) = w^{2^{\Theta(t^{24})}}$. They also provide an efficient algorithm, that, given a wall of size $\Theta(t^{24}(t^2 + w))$, either computes a model of the K_t-minor in G, or returns a set A of at most $O(t^{24})$ vertices, and a flat wall of size w in graph $G \setminus A$.

In this paper we improve their bounds to $f(w, t) = \Theta(t(t + w))$ with $g(t) = t - 5$. We note that this is the best possible bound on $|A|$, since one can construct a graph G containing an arbitrarily large wall and no K_t-minor, such that at least $t - 5$ vertices need to be removed from G in order to obtain a flat wall of size w for any $w > 2$ (see Section 2). For the special case where the maximum vertex degree in G is bounded by D, we show that, if G contains a wall of size $\Omega(Dt(t + w))$, then either G contains a K_t-minor, or there is a flat wall of size w in G. This latter setting naturally arises in algorithms for the Edge-Disjoint Paths problem, with $D \leq 4$. Like the proof of Kawarabayashi et al., our proof is self-contained, except for using the Two-Disjoint-Paths Theorem. We also provide efficient algorithms that return either a model of the K_t-minor, or a vertex set A and a flat wall of size w in $G \setminus A$. We complement our latter result by proving an almost matching lower bound: namely, for all integers $w, t > 1$, there is a graph G, containing a wall of size $\Omega(wt)$, such that the maximum vertex degree in G is 5, and G contains no flat wall of size w, and no K_t-minor.

We now briefly summarize our techniques and compare them to the techniques of Kawarabayashi et al. [KTW12]. The proof of the flat wall theorem in [KTW12] proceeds as follows. Let W be the $(R \times R)$ wall in G. Kawarabayashi et al. start by showing that either there is a collection $\mathcal{P} = \{P_1, \ldots, P_k\}$ of $k = \Omega(t^{12})$ disjoint paths in G, where each path P_i connects a pair of vertices $x_i, y_i \in W$, and is internally disjoint from W, such that the distance between every pair of vertices in set $\{x_i, y_i \mid 1 \leq i \leq k\}$ is large in W; or there is a set A of $O(t^{24})$ vertices, such that, if P is a path in graph $G \setminus A$ connecting a pair of vertices $x, y \in W$, such that P is internally disjoint from W, then the distance between the endpoints of P is small in W. In the former case, the paths in $\mathcal{P}$ are exploited, together with the wall W to find a model of the K_t-minor in G. Assume now that the latter case happens. The wall W is then partitioned into $O(t^{24})$ disjoint horizontal strips of equal height, so at least one of the strips does not contain any vertex of A. Denote this strip by S. Strip S is in turn partitioned into a large number of disjoint walls, where each wall spans a number of consecutive columns of the strip S. They show that either one of the resulting walls contains a large sub-wall that is flat in graph $G \setminus A$; or we can find a model of a K_t-minor in G.

The starting point of our proof is somewhat different. Instead of working with a square $(R \times R)$ wall, it is more convenient for us to work with a wall whose width r is much larger than its height h. In order to achieve this, we start with the $(R \times R)$ wall, and partition it into horizontal strips of height h. We then connect these strips in a snake-like manner to obtain one long wall, of width $\Omega(R^2/h)$ and height h. This strip is partitioned into $\Omega(R^2/h^2)$ disjoint walls of size $(h \times h)$, that we call *basic walls*. Let $\mathcal{B} = (B_1, \ldots, B_N)$ be the resulting sequence of basic walls, for $N = \Omega(R^2/h^2)$. For each such basic wall B_i, we define a core sub-wall B'_i of B_i, obtained from B_i by deleting the top $2t$ and the bottom $2t$ rows. The construction of the long strip and its partition into basic walls and core walls imposes a convenient structure on the wall, that allows us to improve the parameters of the flat wall theorem. Let Γ'_i be the outer boundary of the core wall B'_i. A path P connecting a vertex in $B'_i \setminus \Gamma'_i$ to some vertex of $W \setminus B'_i$, such that P is internally disjoint from W, is called a *bridge for* B'_i. If the other endpoint of P lies in one of the walls B_{i-1}, B_i, B_{i+1}, then we call it a *neighborhood bridge*.

We show that if we can find a collection of $\Omega(t^2)$ disjoint neighborhood bridges incident on distinct core walls, or a collection of $\Omega(t^2)$ disjoint non-neighborhood bridges incident on distinct core walls, then we can find a K_t-minor in G. Our constructions of the K_t-minors are more efficient than those in [KTW12], in that they require a much smaller number of disjoint bridges. This is achieved by exploiting the convenient structure of a long wall partitioned into basic walls, and several new ways to embed a clique minor into G. In a theorem somewhat similar to that of [KTW12], we show that we can either find a collection of $\Omega(t^2)$ disjoint non-neighborhood bridges incident on distinct core walls, or there is a set A of $O(t^2)$ vertices, and a large subset $\mathcal{B}'$ of basic walls, such that for each basic wall $B_i \in \mathcal{B}'$, every bridge for the corresponding core wall B'_i in graph $G \setminus A$ is a neighborhood bridge. In the former case, we use the disjoint non-neighborhood bridges to find a K_t-minor. Assume now that the latter case happens. If many of the walls in $\mathcal{B}'$ have neighborhood bridges incident on their corresponding core walls, then we construct a collection of $\Omega(t^2)$ disjoint neighborhood bridges incident on distinct core walls, which implies that G contains a K_t-minor. Otherwise, for each wall $B_i \in \mathcal{B}'$, we try to find a pair P_i, Q_i of disjoint paths, with P_i connecting the top left corner of B'_i to its bottom right corner, and Q_i connecting its top right corner to its bottom left corner, such that P_i, Q_i are internally disjoint from W. We show that either we can find, for each $B_i \in \mathcal{B}'$, the desired pair (P_i, Q_i) of paths, such that all these paths are disjoint, or one of the walls $B_i \in \mathcal{B}'$ contains a large sub-wall that is flat in $G \setminus A$ (a more careful analysis than the one described here leads to an improved bound of $|A| \leq t-5$). In the former case, we again construct a K_t-minor, by exploiting the paths $\{P_i, Q_i\}_{B_i \in \mathcal{B}'}$, while in the latter case we obtain the desired flat wall in $G \setminus A$. The main difference of our approach from that of [KTW12] is (1) converting the square wall W into a long strip S, which is partitioned into smaller square basic walls, and defining a core wall for each basic wall. Performing this step right at the beginning of the algorithm imposes a convenient structure on the wall W that makes the analysis easier; (2) we propose more different ways to embed a K_t-minor into G, which in turn lead to improved parameters; and (3) careful analysis that allows us to lower $|A|$ from $\Theta(t^2)$ to $t-5$, without increasing the size of the wall we start from.

Organization. We start with preliminaries in Section 2. Since the formal statements of our main results require defining some graph-theoretic notation, these statements can also be found in Section 2. In Sections 3–4 we lay the foundations for proving both upper bounds: in Section 3 we describe several families of graphs such that, if G contain any such graph as a minor, then it must contain a K_t-minor. In Section 4 we describe an algorithm that turns a square $R \times R$ wall into a "long" wall of height h and width R^2/h. This long wall is then partitioned into R^2/h^2 basic walls of size $h \times h$. The basic walls are in turn partitioned into several types, and we show how to handle most of these types. Sections 5 and 6 complete the proofs of the two upper bounds, where Section 5 focuses on the small-degree case, and 6 handles general graphs. We provide the proof of our lower bound in Section 7.

2 Preliminaries and Statements of the Main Theorems

Throughout the paper, we use two parameters: t and w, and our goal is to either find a K_t-minor or a flat wall of size $(w \times w)$. We denote $T = t(t-1)/2$ throughout the paper.

We say that a path P is *internally disjoint* from a set U of vertices, if no vertex of U serves as an inner vertex of P. We say that two paths P, P' are *internally disjoint*, iff for each $v \in V(P) \cap V(P')$, v is an endpoint of both paths.

Given a graph G and three sets A, X, B of vertices of G, we say that X *separates* A from B iff $G \setminus X$ contains no paths from the vertices of $A \setminus X$ to the vertices of $B \setminus X$.

DEFINITION 2.1. *A* separation *in graph G is a pair G_1, G_2 of subgraphs of G, such that $G = G_1 \cup G_2$ and $E(G_1) \cap E(G_2) = \emptyset$. The* order of the separation *is $|V(G_1) \cap V(G_2)|$.*

Notice that if (G_1, G_2) is a separation of G, then there are no edges in G between $V(G_1 \setminus G_2)$ and $V(G_2 \setminus G_1)$.

DEFINITION 2.2. *Given a graph G and a path P in G, we say that P is a 2-path iff every inner vertex of P has degree 2 in G. In other words, P is an induced path in G. We say that P is a* maximal 2-path *iff the degree of each of the two endpoints of P is not 2.*

Minors and Models. We say that a graph H is a *minor* of a graph G, iff H can be obtained from G by a series of edge deletion, vertex deletion, and edge contraction operations. Equivalently, H is a minor of G iff there is a map $f : V(H) \to 2^{V(G)}$ assigning to each vertex $v \in V(H)$ a subset $f(v)$ of vertices of G, such that: (i) for each $v \in V(H)$, the sub-graph of G induced by $f(v)$ is connected; (ii) if $u, v \in V(H)$ and $u \neq v$, then $f(u) \cap f(v) = \emptyset$; and (iii) for each edge $e = (u, v) \in E(H)$, there is an edge in $E(G)$ with one endpoint in $f(v)$ and the other endpoint in $f(u)$. A map f satisfying these conditions is called *a model of H in G*. The following observation follows easily from the definition of minors.

OBSERVATION 2.1. *If H is a minor of G and H' is a minor of H then H' is a minor of G.*

It is sometimes more convenient to use embeddings instead of models for graph minors. A valid embedding of a graph H into a graph G is a map φ, mapping every vertex $v \in V(H)$ to a connected sub-graph $\varphi(v)$ of G, such that, if $u, v \in V(H)$ with $u \neq v$, then $\varphi(v) \cap \varphi(u) = \emptyset$. Each edge $e = (u, v) \in E(H)$ is mapped to a path $\varphi(e)$ in G, such that one endpoint of $\varphi(e)$ belongs to $V(\varphi(v))$, another endpoint to $V(\varphi(u))$, and the path does not contain any other vertices of $\bigcup_{v' \in V(H)} \varphi(v')$. We also require that all paths in $\{\varphi(e) \mid e \in E(H)\}$ are internally disjoint. A valid embedding of H into G can be easily converted into a model of H in G, and can be used to certify that H is a minor of G.

Walls and Grids. In this part we formally define grid graphs and wall graphs. We note that Kawarabayashi et al. [KTW12] provide an excellent overview and intuitive definitions for all terminology needed in the statement of the Flat Wall Theorem. Many of our definitions and explanations in this section follow their paper.

We start with a grid graph. A grid of height h and width r (or an $(h \times r)$-grid), is a graph, whose vertex set is: $\{v(i, j) \mid 1 \leq i \leq h; 1 \leq j \leq r\}$. The edge set consists of two subsets: a set of *horizontal edges* $E_1 = \{(v(i, j), v(i, j+1)) \mid 1 \leq i \leq h; 1 \leq j < r\}$; and a set of *vertical edges* $E_2 = \{(v(i, j), v(i+1, j)) \mid 1 \leq i < h; 1 \leq j \leq r\}$. The sub-graph induced by E_1 consists of h disjoint paths, that we refer to as *the rows of the grid*. The ith row, that we denote by R_i, is the row incident on $v(i, 1)$. Similarly, the sub-graph induced by E_2 consists of r disjoint paths, that we refer to as *the columns of the grid*. The jth column, that is denoted by C_j, is the column starting from $v(1, j)$. Geometrically, we view the rows $R_1, \ldots, R_h$ as ordered from top to bottom, and the columns $C_1, \ldots, C_r$ as ordered left-to-right in the standard drawing of the grid. We say that vertices $v(i, j)$ and $v(i', j')$ of the grid are separated by at least z columns iff $|j - j'| > z$.

We now proceed to define a wall graph W. In order to do so, it is convenient to first define an *elementary wall* graph, that we denote by $\hat{W}$. To construct an elementary wall $\hat{W}$ of height h and width r (or an $(h \times r)$-elementary wall), we start from a grid of height h and width $2r$. Consider some column C_j of the grid, for $1 \leq j \leq r$, and let $e_1^j, e_2^j, \ldots, e_{h-1}^j$ be the edges of C_j, in the order of their appearance on C_j, where e_1^j is incident on $v(1, j)$. If j is odd, then we delete from the graph all edges e_i^j where i is even. If j is even, then we delete from the graph all edges e_i^j where i is odd. We process each column C_j of the grid in this manner, and in the end delete all vertices of degree 1. The resulting graph is an elementary wall of height h and width r, that we denote by $\hat{W}$ (See Figure 1).

Let E'_1 be the set of edges of $\hat{W}$ that correspond to the horizontal edges of the original grid, and let E'_2 be the set of the edges of $\hat{W}$ that correspond to the vertical edges of the original grid, so $E'_1 = E_1, E'_2 \subseteq E_2$. Notice that as before, the sub-graph of $\hat{W}$ induced by E'_1 defines a collection of h node-disjoint paths, that we refer to as the rows of $\hat{W}$. We denote these rows by $R_1, \ldots, R_h$, where for $1 \leq i \leq h$, R_i is incident on $v(i, 1)$. (It will be clear from context whether we talk about the rows of a wall graph or of a grid graph). Let V_1 denote the set of all vertices in the first row of $\hat{W}$, and V_h the set of vertices in the last row of $\hat{W}$. There is a unique set $\mathcal{C}$ of r node-disjoint paths, where each path $C \in \mathcal{C}$ starts at a vertex of V_1, terminates at a vertex of V_h, and is internally disjoint from $V_1 \cup V_h$. We refer to these paths as the columns of $\hat{W}$. We order these columns from left to right, and denote by C_j the jth column in this ordering, for $1 \leq j \leq r$. The sub-graph $Z = R_1 \cup C_1 \cup R_h \cup C_r$ of $\hat{W}$ is a simple cycle, that we call the *outer bondary of* W. We now define the four corners of the wall. The top left corner a is the unique vertex in the intersection of R_1 and C_1; the top right corner b is the unique vertex in the intersection of R_1 and C_r. Similarly, the bottom left and right corners, d and c are defined by $R_h \cap C_1$ and $R_h \cap C_r$, respectively (see Figure 1). All vertices of Z that have degree 2 are called the *pegs* of $\hat{W}$.

We say that a graph W is *a wall of height h and width r*, or an $(h \times r)$-wall, iff it is a subdivision of the elementary wall $\hat{W}$ of height h and width r. Notice that in this case, there is a natural mapping $f : V(\hat{W}) \rightarrow V(W)$, such that for $u \neq v$, $f(u) \neq f(v)$, and for each edge $e = (u, v) \in E(\hat{W})$, there is a path P_e in W with endpoints $f(u), f(v)$, such that all paths $\left\{P_e \mid e \in E(\hat{W})\right\}$ are internally disjoint from each other, and do not contain the vertices of $\left\{f(u') \mid u' \in V(\hat{W})\right\}$ as inner vertices. We call such a mapping *a good* $(\hat{W}, W)$*-mapping*. The corners of W are defined to be the vertices to which the corners of $\hat{W}$ are mapped, and the pegs of W are the vertices to which the pegs of $\hat{W}$ are mapped. Notice that the mapping f is not unique, and so the choice of the corners and the pegs of W is not fixed. For convenience, throughout this paper, the paths P_e of W corresponding to the horizontal edges of $\hat{W}$ are called *blue paths*, and the paths P_e corresponding to the vertical edges of $\hat{W}$ are called *red paths*. For each $1 \leq i \leq h$ and $1 \leq j \leq r$, the ith row of W, R_i, and the jth column of W, C_j, are naturally defined as the paths corresponding (via f) to the ith row and jth column of $\hat{W}$, respectively. A $(w \times w)$-wall is sometimes called *a wall of size w*.

Definition 2.3. *Let W', W be two walls, where W' is a sub-graph of W. We say that W' is a* sub-wall of W *iff every row of W' is a sub-path of a row of W, and every column of W' is a sub-path of a column of W.*

Notice that if a wall W is a sub-division of an elementary wall $\hat{W}$, and we are given some $(\hat{W}, W)$-good mapping $f : V(\hat{W}) \rightarrow V(W)$, then any sub-wall $\hat{W}'$ of $\hat{W}$ naturally defines a sub-wall W' of W: wall W' is the union of all paths P_e for $e \in E(\hat{W}')$. Moreover, since f is fixed, the corners and the pegs of W' are uniquely defined.

We will often work with a special type of sub-walls of a given wall W — sub-walls spanned by contiguous sets of rows and columns of W. We formally define such sub-walls below.

Consider an $(h \times r)$ elementary-wall $\hat{W}$, and let $1 \leq i_1 < i_2 \leq h$ be integers. We define a sub-wall of $\hat{W}$ spanned by rows $(R_{i_1}, \ldots, R_{i_2})$ to be the sub-graph of $\hat{W}$ induced by $\bigcup_{i=i_1}^{i_2} V(R_i)$. Similarly, for integers $1 \leq j_1 < j_2 \leq r$ we define a sub-wall of $\hat{W}$ spanned by columns $(C_{j_1}, \ldots, C_{j_2})$ to be the graph obtained from $\hat{W}$, by deleting all vertices in $\left(\bigcup_{j=1}^{j_1-1} V(C_j)\right) \cup \left(\bigcup_{j=j_2+1}^{r} V(C_j)\right)$, and deleting all vertices whose degree is less than 2 in the resulting graph. The sub-wall $\hat{W}''$ of $\hat{W}$ spanned by rows $(R_{i_1}, \ldots, R_{i_2})$ and columns $(C_{j_1}, \ldots, C_{j_2})$ is computed as follows: let $\hat{W}'$ be sub-wall of $\hat{W}$ spanned by rows $(R_{i_1}, \ldots, R_{i_2})$. Then $\hat{W}''$ is the sub-wall of $\hat{W}'$ spanned by columns $(C_{j_1}, \ldots, C_{j_2})$.

Finally, assume we are given any $(h \times r)$-wall W, the corresponding $(h \times r)$-elementary wall $\hat{W}$ and a $(\hat{W}, W)$-good mapping $f : V(\hat{W}) \rightarrow V(W)$. For integers $1 \leq i_1 < i_2 \leq h$, and $1 \leq j_1 < j_2 \leq r$, we define the sub-wall W' of W spanned by rows $(R_{i_1}, \ldots, R_{i_2})$ and columns $(C_{j_1}, \ldots, C_{j_2})$, as follows. Let $\hat{W}'$ be the sub-wall of $\hat{W}$ spanned by rows $(R_{i_1}, \ldots, R_{i_2})$ and columns $(C_{j_1}, \ldots, C_{j_2})$. We then let W' be the unique sub-wall of W corresponding to $\hat{W}'$ via the mapping f. That is, W' is the union of all paths P_e for $e \in E(\hat{W}')$. As observed before, since the mapping f is fixed, the corners and the pegs of W' are uniquely defined. Sub-walls of W spanned by sets of consecutive rows, and sub-walls spanned by sets of consecutive columns are defined similarly.

From our definition of an elementary wall, it is clear that the $(h \times 2r)$-grid contains the $(h \times r)$-elementary wall as a minor. It is also easy to see that an $(h \times r)$-wall W contains the $(h \times r)$-grid G as a minor: let $\hat{W}$ be the $(h \times r)$-elementary wall, and assume that we are given some $(\hat{W}, W)$-good mapping $f : V(\hat{W}) \rightarrow V(W)$. Clearly, $\hat{W}$ is a minor of W. For every $1 \leq i \leq h$, $1 \leq j \leq r$, let $P(i, j) = R_i \cap C_j$, where R_i and C_j are the ith row and the jth column of $\hat{W}$, respectively. We contract all edges in $P(i, j)$. Once we process all pairs R_i, C_j in this manner, we obtain the $(h \times r)$-grid $\tilde{G}$. We call $\tilde{G}$ *a contraction* of W. Notice that if the mapping $f : V(\hat{W}) \rightarrow V(W)$ is fixed, then this contraction is uniquely defined, and so is the model of $\tilde{G}$ in W.

Linkedness. We now turn to define the notion of t-linkedness that we use extensively in our proof.

Definition 2.4. *For any integer $t > 0$, we say that two disjoint sets X, Y of vertices of are t-linked in graph G, iff for any pair $X' \subseteq X$, $Y' \subseteq Y$ of vertex subsets, with $|X'| = |Y'| \le t$, there is a set of $|X'|$ node-disjoint paths in graph G, connecting the vertices of X' to the vertices of Y'.*

A useful feature of grid graphs is that the sets of vertices in the first and the last columns of the grid are t-linked, as long as t is no larger than the smaller of the dimensions of the grid. We show this in the following claim, whose proof is omitted due to lack of space.

Claim 2.1. *Let G be an $(h \times r)$ grid, $t \le \min\{h, r\}$ an integer, X the set of all vertices on the first column of G and Y the set of all vertices on the last column of G. Then X and Y are t-linked in G.*

A similar claim holds for wall graphs, except that we need to be more careful in defining the sets X and Y of vertices.

Claim 2.2. *Let $\hat{W}$ be an $(h \times r)$-elementary wall, $t \le \min\{h, r\}$ a parameter, X a set of vertices lying in the first column of G and Y a set of vertices lying in the last column of G, such that for each row R_i of $\hat{W}$, $|X \cap R_i| \le 1$ and $|Y \cap R_i| \le 1$. Then X and Y are t-linked in G.*

C-Reductions and Flat Walls.

Definition 2.5. *Let G be a graph, $X \subseteq V(G)$, and let (A, B) be a separation of G of order at most 3 with $X \subseteq A$. Moreover, assume that the vertices of $A \cap B$ are connected in B. Let H be the graph obtained from $G[A]$ by adding an edge connecting every pair of vertices in $A \cap B$. We say that H is an elementary X-reduction in G, determined by (A, B). We say that a graph J is an X-reduction of G if it can be obtained from G by a series of elementary X-reductions.*

We need a definition of C-flat graphs. Intuitively, let G be any graph, and let C be any simple cycle of G. Suppose there is some C-reduction H of G, such that H is a planar graph, and there is a drawing of H in which C bounds its outer face. Then we say that G is C-flat. Following is an equivalent way to define C-flat graphs, due to [KTW12], which is somewhat more convenient to work with.

Definition 2.6. *Let G be a graph, and let C be a cycle in G. We say that G is C-flat if there exist subgraphs $G_0, G_1, \ldots, G_k$ of G, and a plane graph $\tilde{G}$, such that:*

- *$G = G_0 \cup G_1 \cup \cdots \cup G_k$, and the graphs $G_0, G_1, \ldots, G_k$ are pairwise edge-disjoint;*
- *C is a subgraph of G_0.*
- *G_0 is a subgraph of $\tilde{G}$, with $V(\tilde{G}) = V(G_0)$. Moreover, $\tilde{G}$ is a plane graph, and the cycle C bounds its outer face;*
- *For all $1 \le i \le k$, $|V(G_i) \cap V(G_0)| \le 3$.*
  - *If $|V(G_i) \cap V(G_0)| = 2$, then u and v are adjacent in $\tilde{G}$;*
  - *If $V(G_i) \cap V(G_0) = \{u, v, w\}$, then some finite face of $\tilde{G}$ is incident with u, v, w and no other vertex;*
- *For all $1 \le i \ne j \le k$, $V(G_i) \cap V(G_j) \subseteq V(G_0)$.*

We are now ready to define a flat wall.

Definition 2.7. *Let G be a graph, and let W be a wall in G with outer boundary D. Suppose there is a separation (A, B) of G, such that $A \cap B \subseteq V(D)$, $V(W) \subseteq B$, and there is a choice of pegs of W, such that every peg belongs to A. If some $A \cap B$-reduction of $G[B]$ can be drawn in a disc with the vertices of $A \cap B$ drawn on the boundary of the disc in the order determined by D, then we say that the wall W is flat in G.*

Statements of the Main Theorems. We need one more definition in order to state our main theorems. Let W be a wall in some graph G, and assume that G contains a K_t-minor. Recall that a model of the K_t-minor in G maps each vertex $v \in V(K_t)$ to a subset $f(v)$ of vertices of G. We say that the K_t-minor is *grasped* by the wall W iff for each $v \in V(K_t)$, $f(v)$ intersects at least t rows of W, or at least t columns of W. We will use the following simple observation.

Observation 2.2. *Let W be an $(h \times r)$-wall in a graph H, and $\tilde{G}$ an $(h \times r)$-grid, such that $\tilde{G}$ is a contraction of W. Suppose we are given a model $f(\cdot)$ of a K_t-minor in $\tilde{G}$, such that for each $v \in V(K_t)$, $f(v)$ intersects at least t rows or at least t columns of $\tilde{G}$. Then there is a model of K_t in H grasped by W.*

The observation follows from the fact that for each row R_i of $\tilde{G}$, every vertex on R_i is mapped by the contraction to a set of vertices of H contained in the ith row of W, and the same holds for the columns of $\tilde{G}$.

We are now ready to state our main theorems. Our first theorem is a slightly weaker version of the flat wall theorem, in that it is mostly interesting for graphs whose maximum vertex degree is relatively small. Such graphs arise, for example, in edge-disjoint routing problems, and the guarantees given by this theorem are somewhat better than the guarantees given by the stronger flat wall theorem that appears below, as we do not need to deal with apex vertices.

THEOREM 2.1. *Let G be any graph with maximum vertex degree D, let $w, t > 1$ be integers, set $T = t(t-1)/2$, and let $R = (w + 4t)\left(2 + \left\lceil\sqrt{8D^2(10T+6)+14T+8}\right\rceil\right) = \Theta(Dt(w+t))$. Then there is an algorithm, that, given any $(R \times R)$-wall $W \subseteq G$, either computes a model of a K_t-minor grasped by W in G, or returns a sub-wall W^* of W of size at least $(w \times w)$, such that W^* is a flat wall in G. The running time of the algorithm is polynomial in $|V(G)|, D, w$ and t.*

THEOREM 2.2. *Let G be any graph, let $w, t > 1$ be integers, set $T = t(t-1)/2$, and let $R = (w + 4t)\left(2 + \left\lceil\sqrt{500T+200}\right\rceil\right) = \Theta(t(w+t))$. Then there is an algorithm, that, given any $(R \times R)$-wall $W \subseteq G$, either finds a model of a K_t-minor grasped by W in G, or returns a set A of at most $t-5$ vertices, and a sub-wall W^* of W of size at least $(w \times w)$, such that $V(W^*) \cap A = \emptyset$ and W^* is a flat wall in $G \setminus A$. The running time of the algorithm is polynomial in $|V(G)|, w$ and t.*

Observe that the bound of $t-5$ on $|A|$ is the best possible. Indeed, let G be the graph obtained from an $(R \times R)$-elementary wall W (for any value R), by adding a set $A = \{a_1, \ldots, a_{t-5}\}$ of new vertices, and connecting every vertex of A to every vertex of W. Clearly, in order to obtain a flat wall of size $(w \times w)$ for any $w > 2$ in G, we need to delete all vertices of A from it. Assume for contradiction that G contains a model f of a K_t-minor. Let $\mathcal{C} = \{f(v) \mid v \in V(K_t)\}$ be the sets of vertices of G to which the vertices of K_t are mapped. Then at most $t-5$ sets in $\mathcal{C}$ may contain vertices of A. So there are at least 5 sets $S_1, \ldots, S_5 \in \mathcal{C}$ of vertices, where for $1 \leq i \leq 5$, $S_i \cap A = \emptyset$. But then sets $S_1, \ldots, S_5$ define a model of a K_5-minor in graph W, and since W is planar, this is impossible.

Our lower bound is summarized in the following theorem.

THEOREM 2.3. *For all integers $w, t > 1$, there is a graph G, containing a wall of size $\Omega(wt)$, such that G does not contain a flat wall of size w, and it does not contain a K_t-minor. The maximum vertex degree of G is 5.*

A C-cross and a Wall-Cross. Suppose we are given a graph G and a cycle C in G. A *C-cross* in G is a pair P_1 and P_2 of disjoint paths, with ends s_1, t_1 and s_2, t_2, respectively, such that s_1, s_2, t_1, t_2 occur in this order on C, and no vertex of C serves as an inner vertex of P_1 or P_2. The next theorem follows from [Jun70, RS90, Sey06, Shi80, Tho80, KTW12].

THEOREM 2.4. *Let G be a graph and let C be a cycle in G. Then the following conditions are equivalent:*

- *G has no C-cross;*
- *Some C-reduction of G can be drawn in the plane with C as a boundary of the outer face;*
- *G is C-flat.*

Moreover, there is an efficient algorithm, that either computes a C-cross in G, or returns the subgraphs $G_0, G_1, \ldots, G_k$ of G and the plane graph $\tilde{G}$, certifying that G is C-flat.

We will be extensively using a special type of cross, connecting the corners of a wall. For brevity of notation, we define it below, and we call it a *wall-cross*.

DEFINITION 2.8. *Let G be any graph, and W an $(h \times r)$-wall in G. Let $\hat{W}$ be the corresponding $(h \times r)$-elementary wall, and assume that we are given a $(\hat{W}, W)$-good mapping $f : V(\hat{W}) \to V(W)$. Let a, b, c, d be the four corners of W (whose choice is fixed given f), appearing on the boundary of W in this order. A wall-cross for W is a pair P_1, P_2 of disjoint paths, where P_1 connects a to c, and P_2 connects b to d.*

Assume that we are given any pair u, v of vertices of a wall W. We say that u and v are separated by a column C_j of W, iff $u, v \notin V(C_j)$, and $V(C_j)$ separates u from v in graph W. Similarly, we say that u, v are separated by a row R_i of W, iff $u, v \notin V(R_i)$, and $V(R_i)$ separates u from v in W. We will repeatedly use the following simple theorem, whose proof is omitted from this extended abstract and can be found in the full version of the paper.

THEOREM 2.5. *Assume that we are given a wall W of height $h \geq 5$ and width $r \geq 5$, with corners a, b, c, d appearing on the boundary of W in this order. Let*

$u, v \in V(W)$ be any pair of vertices, such that one of the following holds: either (1) neither u nor v lie on the boundary of W and they are separated by some row or some column of W; or (2) u lies on the boundary of W, and v lies in the sub-wall of W spanned by rows $(R_3, \ldots, R_{h-2})$ and columns $\{C_3, \ldots, C_{r-2}\}$. Let W' be the graph obtained from W by adding the edge (u, v) to it. Then there is a wall-cross for W in W'.

3 Some Useful Graphs

In this section we construct a graph H^*, and define three families of graphs: $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, such that, if G contains H^* or one of the graphs in $\mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$ as a minor, then it contains a K_t-minor.

3.1 Graph H^*. We start with a grid containing $2t$ rows, that we denote by $R_1, \ldots, R_{2t}$, and $Tt+1$ columns, denoted by $C_1, \ldots, C_{Tt+1}$. The vertex lying at the intersection of row R_i and column C_j is denoted by $v(i, j)$.

Consider the vertices $u_i = v(t, ti)$ for $1 \leq i \leq T$ (so these are vertices roughly in the middle row of the grid, spaced t apart horizontally). For each such vertex u_i, let L_i be the cell of the grid for which u_i is the upper left corner. We add two diagonals to this cell, that is, two edges: $e_i = (v(t, ti), v(t+1, ti+1))$, and $e_{i+1} = (v(t+1, ti), v(t, ti+1))$. We call these edges *cross edges*. This completes the definition of the graph H^*. Let J_1 be the sub-graph of H^*, induced by the set $V(C_1) \cup \cdots V(C_t)$ of vertices. The proof of the following lemma is omitted from this extended abstract.

LEMMA 3.1. *Let G be any graph and let W be a wall in G. Assume that G contains H^* as a minor. Then G contains a K_t-minor. Moreover, if J_1 is a contraction of some sub-wall of W, then there is a model of K_t grasped by W in G, and this model can be found efficiently given a model of H^* in G.*

3.2 Graph Families $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$. In this section, we define three graph families $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$. We will show that if G contains a graph from any of these families as a minor, then it contains a K_t-minor. Before we proceed to define these families of graphs, we need a few definitions.

Let G' be the $(h \times r)$-grid. As with walls, we define sub-grids of G' spanned by subsets of rows and columns of G'. For any consecutive subset $\mathcal{R}'$ of the rows of G', the sub-grid of G' spanned by $\mathcal{R}'$ is $G'[S]$, where S contains all vertices $v(i, j)$ with $R_i \in \mathcal{R}'$. Similarly, given any consecutive subset $\mathcal{R}'$ of the rows of G', and a consecutive subset $\mathcal{C}'$ of the columns of G', the sub-grid of G spanned by the rows in $\mathcal{R}'$ and the columns in $\mathcal{C}'$ is $G[S']$, where S' contains all vertices $v(i, j)$ with $R_i \in \mathcal{R}'$ and $C_j \in \mathcal{C}'$.

Assume now that we are given two vertices $v(i, j)$ and $v(i', j')$ of the grid G', where $j \leq j'$. We say that $v(i, j)$ and $v(i', j')$ are *separated* by column $C_{j''}$ of the grid iff $j < j'' < j'$. We say that they are separated by x columns of G' iff at least x distinct columns $C_{j''}$ separate $v(i, j)$ from $v(i', j')$, or, equivalently, $j' - j > x$.

Graph Family $\mathcal{H}_1$. A graph H belongs to the family $\mathcal{H}_1$ iff H is the union of the $(h \times r)$ grid G', where $h > 2t$, and a set E' of T edges, such that the following additional conditions hold. Let G_1 be the sub-grid of G' spanned by the top t rows, G_2 the sub-grid of G' spanned by the bottom t rows, and G_3 the sub-grid of G' spanned by the remaining rows. Let X be the set of all endpoints of the edges in E'. Then the following conditions must hold:

- $X \subseteq V(G_3)$, and $|X| = 2T$, so all edges in E' have distinct endpoints.
- Every pair of vertices in X is separated by at least $t+2$ columns, and no vertex of X belongs to the first column of G'.

Let B_1 be the sub-grid of G' spanned by the first t columns of G' and all rows of G'. The proof of the following theorem is omitted from this extended abstract.

THEOREM 3.1. *Let G be any graph, and assume that it contains a graph $H \in \mathcal{H}_1$ as a minor. Then G contains a K_t-minor. Moreover, if G contains a wall W, and B_1 is a contraction of a sub-wall of W, then G contains a model of a K_t-minor grasped by W, and this model can be found efficiently given a model of H in G.*

Graph Family $\mathcal{H}_2$. A graph H belongs to the family $\mathcal{H}_2$ iff H is the union of the $(h \times r)$ grid G', where $h > 4t$, and a set E' of $2T+2$ edges, and the following conditions hold. Let G_1 be the sub-grid of G' spanned by the top t rows, G_2 the sub-grid of G' spanned by the bottom t rows, and G_3 the sub-grid of G' spanned by rows $\{R_{2t+1}, \ldots, R_{h-2t}\}$. We assume that $E' = \{e_1, \ldots, e_{2T+2}\}$, and for each $1 \leq i \leq 2T+2$, the endpoints of e_i are labeled as x_i and y_i. Let $X = \{x_i \mid 1 \leq i \leq 2T+2\}$, and $Y = \{y_i \mid 1 \leq i \leq 2T+2\}$. Then the following conditions must hold:

- $X \cup Y$ contains $4T+4$ distinct vertices.
- $X \subseteq V(G_3)$, and every pair of vertices in X is separated by at least $t+2$ columns.
- $Y \subseteq V(G_1) \cup V(G_2)$.

Let B_1 be the sub-grid of G' spanned by the first t columns of G' and all rows of G'. The proof of the following theorem is omitted from this extended abstract.

THEOREM 3.2. *Let G be any graph, that contains a graph $H \in \mathcal{H}_2$ as a minor. Then G contains a K_t-minor. Moreover, if G contains a wall W, and B_1 is a contraction of a sub-wall of W, then G contains a model of a K_t-minor grasped by W, and this model can be found efficiently given a model of H in G.*

Graph Family $\mathcal{H}_3$. A graph H belongs to the family $\mathcal{H}_3$ iff H is the union of the $(h \times r)$ grid G', where $h > 4t$, and a set E' of $10T+6$ edges, and the following conditions hold. Let G_1 be the sub-grid of G' spanned by the top t rows, G_2 the sub-grid of G' spanned by the bottom t rows, and G_3 the sub-grid of G' spanned by rows $\{R_{2t+1}, \ldots, R_{h-2t}\}$. We assume that $E' = \{e_1, \ldots, e_{10T+6}\}$, and for each $1 \leq i \leq 10T+6$, the endpoints of e_i are labeled as x_i and y_i. Let $X = \{x_i \mid 1 \leq i \leq 10T+6\}$, and $Y = \{y_i \mid 1 \leq i \leq 10T+6\}$. Then the following conditions must hold:

- $X \cup Y$ contains $20T+12$ distinct vertices.
- $X \subseteq V(G_3)$, and every pair of vertices in X is separated by at least $2t+2$ columns in G'.
- For each $1 \leq i \leq 10T+6$, x_i and y_i are separated by at least $t+1$ columns in G'.
- No vertex of $X \cup Y$ lies in the first t columns, or in the last column of G'.

Let J_1 be the sub-grid of G' spanned by the first $t+1$ columns of G' and all rows of G'. The proof of the following theorem is omitted from this extended abstract.

THEOREM 3.3. *Let G be any graph, that contains a graph $H \in \mathcal{H}_3$ as a minor. Then G contains a K_t-minor. Moreover, if G contains a wall W, and J_1 is a contraction of a sub-wall of W, then G contains a model of a K_t-minor grasped by W, and this model can be found efficiently given a model of H in G.*

4 Chain of Walls, Bridges, Core Walls, and Wall Types

Our starting point is a combinatorial object that we call a chain of basic walls.

DEFINITION 4.1. *A* chain $(\mathcal{B}, \mathcal{P})$ of N basic walls of height z *consists of:*

- *A collection $\mathcal{B}$ of N disjoint walls $B_1, \ldots, B_N$, that we call* basic walls, *where each wall B_i has height z and width at least z.*
- *A set $\mathcal{P} = \bigcup_{j=1}^{N-1} \mathcal{P}_j$ of disjoint paths, where for each $1 \leq j < N$, $\mathcal{P}_j = \left\{P_1^j, \ldots, P_z^j\right\}$ is a set of z paths, connecting the pegs of B_j lying in the last column of B_j to the pegs of B_{j+1} lying in the first column of B_{j+1}, and for $1 \leq i \leq z$, P_i^j connects a vertex in the ith row of B_j to a vertex in the ith row of B_{j+1}. Moreover, the paths in $\mathcal{P}$ do not contain the vertices of $\bigcup_{j'=1}^{N} V(B_{j'})$ as inner vertices.*

We denote by $W'(\mathcal{B}, \mathcal{P})$ the corresponding graph $\left(\bigcup_{j=1}^{N} \mathcal{B}_j\right) \cup \left(\bigcup_{j=1}^{N-1} \mathcal{P}_j\right)$.

Observe that graph $W' = W'(\mathcal{B}, \mathcal{P})$ is a wall of height z and width at least Nz. We will always assume that we are given some fixed choice of the four corners (a, b, c, d) of the wall W', that appear along the boundary of W' in this order clock-wise, and a is the top left corner of W'. Therefore, for each basic wall B_i, the four corners of B_i are also fixed, and are denoted by a_i, b_i, c_i, d_i, where a_i is the top left corner, and the four corners appear in this order clock-wise along the boundary of B_i. We need the following theorem.

THEOREM 4.1. *For any integers $N, z \geq 2$, given a wall W of size $(Nz \times Nz)$, there is an efficient algorithm to construct a chain $(\mathcal{B}, \mathcal{P})$ of $N(N-2)$ basic walls of height z, such that $W'(\mathcal{B}, \mathcal{P})$ is a sub-graph of W, and each basic wall $B \in \mathcal{B}$ is a sub-wall of W.*

The proof of the theorem can be found in the full version of the paper; we provide an informal overview here. We sketch the proof for the case where W is an elementary wall; if the input wall W is not elementary, then we first build a chain of walls in the corresponding elementary wall $\hat{W}$, and then use it to define a chain of walls for W in a natural manner. For each $1 \leq i < N$, we delete all edges of W connecting row iz to row $iz+1$, thus obtaining a partition of W into N horizontal strips

$S_1, \ldots, S_N$, where each strip is a wall of height z and width Nz. Each such strip is in turn partitioned into N disjoint $(z \times z)$-walls, that we call basic walls. Let $\mathcal{B}$ be the corresponding collection of basic walls. For each horizontal strip S_i, we then discard the first and the last basic wall of S_i from $\mathcal{B}$, and we use these basic walls to connect the horizontal strips together, in a snake-like fashion, to obtain one wall of height z and width $N(N-2)z$. This final wall, together with the set $\mathcal{B}$ of basic walls defines the desired chain of walls.

Let G be any graph, $(\mathcal{B}, \mathcal{P})$ a chain of N basic walls of height z in G, and let $W' = W'(\mathcal{B}, \mathcal{P})$ be the corresponding sub-graph of G. Let $\mathcal{C}$ be the set of all connected components of $G \setminus V(W')$. We say that a component $F \in \mathcal{C}$ *touches* a vertex $v \in V(W')$ iff G contains an edge from a vertex of F to v.

Given an integer parameter $1 \le \tau < z/2$, for each $1 \le i \le N$, we define a *τ-core sub-wall* B'_i of B_i, as follows. Wall B'_i is the sub-wall of B_i spanned by rows $(R_\tau, \ldots, R_{z-\tau+1})$ and all columns of B_i (see Figure 2). The boundary of the τ-core wall B'_i is denoted by Γ'_i, and its four corners are denoted by a'_i, b'_i, c'_i, d'_i. We will assume throughout this section that the value of τ is fixed, and we will sometimes refer to the τ-core walls simply as core walls.

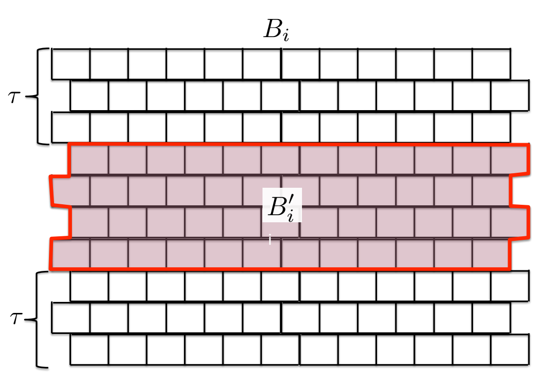

Figure 2: Graphs B_i and B'_i.

DEFINITION 4.2. *A* bridge *incident on a core wall B'_i is one of the following: either an edge with one endpoint in $V(B'_i \setminus \Gamma'_i)$, and another in $V(W' \setminus B'_i)$; or a connected component $F \in \mathcal{C}$, that touches a vertex of $V(B'_i \setminus \Gamma'_i)$, and a vertex of $V(W' \setminus B'_i)$.*

For $1 < i < N$, we define the neighborhood of B_i as follows: $\mathcal{N}(B_i) = V(B_{i-1} \cup \mathcal{P}_{i-1} \cup B_i \cup \mathcal{P}_i \cup B_{i+1})$. We say that a bridge F incident on the core wall B'_i is a *neighborhood bridge* for B'_i iff either F is an edge whose both endpoints lie in $\mathcal{N}(B_i)$, or F is a connected component of $\mathcal{C}$, such that all vertices of W' that F touches belong to $\mathcal{N}(B_i)$. Otherwise, we say that F is a *non-neighborhood* bridge for B'_i.

For $1 < i < N$, we say that the core wall B'_i is a type-1 wall iff there is at least one bridge F incident on B'_i, such that F is a neighborhood bridge for B'_i. We say that it is a type-2 wall if it is not a type-1 wall, and at least one bridge is incident on B'_i. Therefore, if B'_i is a type-2 wall, then at least one bridge F incident on B'_i is a non-neighborhood bridge for B'_i.

Assume now that B'_i is not type-1 and not type-2 wall. Then no bridge is incident on B'_i, so graph $G \setminus \Gamma'_i$ consists of at least two connected components, with one of them containing $B'_i \setminus \Gamma'_i$. Therefore, there is a separation (X, Y) of G, with $B'_i \subseteq X$, $X \cap Y \subseteq \Gamma'_i$, and for each $1 \le j \le N$ with $j \neq i$, $B_j \subseteq Y$. Recall that the corners a'_i, b'_i, c'_i, d'_i of the wall B'_i are fixed. We assume that they appear on Γ'_i in this order clockwise, with a'_i being the top left corner. If graph X contains a wall-cross for B'_i (that is, a pair of disjoint paths connecting a'_i to c'_i and b'_i to d'_i), then we say that wall B'_i is of type 3. Otherwise, it is of type 4. Notice that given W', for each $1 < i < N$, we can efficiently determine what type wall B'_i belongs to, and if it is a type-3 wall, then we can find the corresponding wall-cross efficiently.

The proofs of both Theorems 2.1 and 2.2 proceed in a similar way: we start with a wall W of an appropriate size, and apply Theorem 4.1 to obtain a chain of walls $(\mathcal{B}, \mathcal{P})$ with some parameters z and N. We then show that if, for any of the basic walls in the chain, the corresponding τ-core wall is of type 4, then G contains a flat sub-wall of W of size $((z - 2\tau) \times (z - 2\tau))$. Therefore, with an appropriate choice of z and τ, we can assume that all τ-core walls are of types 1, 2 or 3. For each one of these three types, we show that if there are many τ-core walls of that type, then G contains a K_t-minor grasped by W. Therefore, large parts of the proofs of both theorems are similar, and only differ in the specific parameters we choose. In the rest of this section we formally state and prove theorems that allow us to handle walls of each one of the four types. The statements of the theorems are generic enough so we can apply them in the different settings with the different choices of the parameters that we need.

We start by observing that if at least one basic wall of $\mathcal{B}$ is of type 4, then G contains a flat wall of size $((z - 2\tau) \times (z - 2\tau))$. The proof of the following lemma is omitted here and appears in the full version of the paper.

LEMMA 4.1. *Let G be any connected graph, B a wall* *of size $((z+2)\times(z+2))$ in G, Γ an outer boundary of B, and a,b,c,d the corners of B appearing on Γ in this order circularly. Assume further that there is a separation (X,Y) of G, with $B\subseteq X$ and $X\cap Y\subseteq\Gamma$, such that X does not contain a wall-cross for B. Then there is an efficient algorithm to find a flat wall B' of size $(z\times z)$ in G, such that B' is a sub-wall of B.*

We then obtain the following immediate corollary:

COROLLARY 4.1. *Let G be any connected graph, W a wall in G, $(\mathcal{B},\mathcal{P})$ a chain of N walls of height at least z in G, and $\tau<z/2$ some integer. Assume that for some $1<i<N$, the τ-core wall B'_i is a type-4 wall. Then there is an efficient algorithm to find a flat wall of size $((z-2\tau)\times(z-2\tau))$ in G. Moreover, if B_i is a sub-wall of W, then so is the flat wall.*

4.1 Type-3 Core Walls. We use the following theorem to handle type-3 walls.

THEOREM 4.2. *Let G be a connected graph, $\tau>t$ an integer, W a wall in G, and $(\mathcal{B},\mathcal{P})$ a chain of N walls of height $z>2\tau$ in G. If the number of τ-core walls B'_i of type 3 with $1<i<N$ in the chain of walls is at least $2T$, then G contains a K_t-minor. Moreover, if $B_1\in\mathcal{B}$ is a sub-wall of W, then G contains a model of a K_t-minor that is grasped by W, and it can be found efficiently given $(\mathcal{B},\mathcal{P})$.*

In order to prove the theorem, we show that G contains the graph H^*, defined in Section 3, as a minor. The theorem then follows from Lemma 3.1 and Observation 2.1. The proof can be found in the full version of the paper.

4.2 Type-1 Core Walls In this section we take care of type-1 core walls, by proving the following theorem.

THEOREM 4.3. *Let G be a connected graph, $\tau\geq 2t$ an integer, W a wall in G, and $(\mathcal{B},\mathcal{P})$ a chain of N walls of height $z>2\tau$ in G. If the number of τ-core walls $B'_i\in\mathcal{B}$ of type 1 with $1<i<N$ is at least $12T+6$, then G contains a K_t-minor. Moreover, if $B_1\in\mathcal{B}$ is a sub-wall of W, then G contains a model of a K_t-minor grasped by W, and it can be found efficiently, given $(\mathcal{B},\mathcal{P})$.*

Proof. Let $\mathcal{S}$ be the set of τ-core walls B'_i of type 1, with $1<i<N$. We select a subset $\mathcal{S}'\subseteq\mathcal{S}$ of $4T+2$ core walls, such that, if we denote $\mathcal{S}'=\left\{B'_{i_1},B'_{i_2},\ldots,B'_{4T+2}\right\}$, where $1<i_1<i_2<\ldots<i_{4T+2}<N$, then $i_1>2$, and for all $1\leq r<4T+2$, $i_{r+1}\geq i_r+3$. In other words, $B'_1,B'_2,B'_N\notin\mathcal{S}'$, and every consecutive pair of core walls is separated by at least two walls. In order to construct $\mathcal{S}'$, we can use a simple greedy algorithm: order the walls in $\mathcal{S}$ in the ascending order of their indices; add to $\mathcal{S}'$ all walls whose indices are 0 modulo 3 in this ordering; delete walls from $\mathcal{S}'$ as necessary until $|\mathcal{S}'|=4T+2$ holds.

Consider now some wall $B'_{i_r}\in\mathcal{S}'$. Recall that there is at least one neighborhood bridge F_r incident on B'_{i_r}. This bridge must contain a path P_{i_r}, connecting a vertex of $B'_{i_r}\setminus\Gamma'_{i_r}$ to a vertex of $\mathcal{N}(B_{i_r})\setminus B'_{i_r}$, such that P_{i_r} does not contain any vertices of W' as internal vertices. We denote the endpoint of P_{i_r} that belongs to $B'_{i_r}\setminus\Gamma'_{i_r}$ by x_{i_r}, and its other endpoint by y_{i_r}. Notice that since $\tau\geq 2t$, x_{i_r} cannot belong to the sub-walls of W' spanned by the top $2t$ or the bottom $2t$ rows of W'. All paths $\{P_i\}_{B'_i\in\mathcal{S}'}$ are completely disjoint from each other, since for $r\neq r'$, $|i_r-i_{r'}|\geq 3$, so the neighborhood bridges $F_{i_r},F_{i_{r'}}$ are disjoint.

Let $\mathcal{R}_1$ be the set of the top t rows of W', $\mathcal{R}_2$ the set of the bottom t rows of W', and $\mathcal{R}_3$ the set of all remaining $z-2t$ rows of W'. For $1\leq j\leq 3$, let W_j be the sub-wall of W' spanned by the rows in $\mathcal{R}_i$ and all columns of W'. We partition $\mathcal{S}'$ into two subsets, $\mathcal{S}_1,\mathcal{S}_2$, where $\mathcal{S}_1$ contains all walls $B'_{i_r}\in\mathcal{S}'$ with $y_{i_r}\in V(W_1\cup W_2)$, and $\mathcal{S}_2$ containing all remaining walls. The following two lemmas will finish the proof.

LEMMA 4.2. *If $|\mathcal{S}_1|\geq 2T+2$, then G contains a K_t-minor. Moreover, if B_1 is a sub-wall of W, then G contains a model of K_t-minor grasped by W, and it can be found efficiently, given $(\mathcal{B},\mathcal{P})$.*

The proof proceeds by showing that graph G contains a graph $H\in\mathcal{H}_2$ as a minor, and then invokes Theorem 3.2. The formal proof can be found in the full version of the paper.

LEMMA 4.3. *If $|\mathcal{S}_2|\geq 2T$, then G contains a K_t-minor. Moreover, if B_1 is a sub-wall of W, then G contains a model of a K_t-minor grasped by W, and it can be found efficiently, given $(\mathcal{B},\mathcal{P})$.*

Proof. The idea of the proof is to define a different partition $\mathcal{B}'$ of W' into basic walls, such that for each original basic wall B_i with $B'_i\in\mathcal{S}_2$, there is a basic wall $\tilde{B}_i\in\mathcal{B}'$ with $\mathcal{N}(B_i)\subseteq\tilde{B}_i$. Let $\tilde{B}'_i$ be the $(t+1)$-core sub-wall of $\tilde{B}_i$, and let Z_i be the graph $\tilde{B}'_i\cup P_i$. We show that Z_i contains a wall-cross for $\tilde{B}'_i$, and so we reduce the problem to the case where the number of $(t+1)$-core walls of type 3 is at least $2T$. Applying Theorem 4.2 then finishes the proof.

Formally, for each wall $B'_i \in \mathcal{S}_2$, let i_1 be the index of the first column of W' whose vertices are contained in $\mathcal{N}(B_i)$, and let i_2 be the index of the last column of W' whose vertices are contained in $\mathcal{N}(B_i)$. We define a set $\mathcal{C}_i$ of consecutive columns of W' to contain all columns starting from C_{i_1} and ending with C_{i_2}. Let $\tilde{B}_i$ be the sub-wall of W' spanned by columns in $\mathcal{C}_i$ and all rows of W'. Notice that for $B'_i, B'_j \in \mathcal{S}_2$, if $i \neq j$, then $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ due to the choice of $\mathcal{S}'$. Let $\mathcal{C}^*$ be the set of the first t columns of W'. We define one additional basic wall $\tilde{B}^*$ to be the sub-wall of W' spanned by the columns in $\mathcal{C}^*$ and all rows of W'. Let $\mathcal{B}'$ be the set of these new basic walls.

Let W'' be the sub-wall of W' obtained by taking the union of all the columns of W' that are contained in the walls in $\mathcal{B}'$, and all the rows of W'. Then W'' and the walls in $\mathcal{B}'$ define a chain of $2T$ walls of height z. Let G' be the union of W'' with all paths P_i for $B'_i \in \mathcal{S}_2$.

For convenience, for each $B'_i \in \mathcal{S}_2$, we re-name the path P_i to be $P(\tilde{B})$ where $\tilde{B} \in \mathcal{B}'$ is the basic wall containing B'_i.

For each $\tilde{B} \in \mathcal{B}'$, where $\tilde{B} \neq \tilde{B}^*$, let $\tilde{B}'$ be the $(t+1)$-core sub-wall of $\tilde{B}$, and let $G(\tilde{B})$ be the union of $\tilde{B}$ with the path $P(\tilde{B})$ (whose both endpoints must be contained in $\tilde{B}'$). Notice that the endpoints of $P(\tilde{B})$ are separated by at least one row or at least one column, as one endpoint of $P(\tilde{B})$ belongs to $B'_j \setminus \Gamma'_j$ and the other to $\mathcal{N}(B_j) \setminus V(B'_j)$ for the corresponding $2t$-core wall $B'_j \in \mathcal{S}_2$. From Theorem 2.5, graph $G(\tilde{B})$ contains a wall-cross for $\tilde{B}'$. Therefore, the chain of walls defined by W'' and $\mathcal{B}'$ contains $2T$ $(t+1)$-core walls, that are type-3 walls in graph G'. From Theorem 4.2, G' must contain a K_t-minor. Moreover, if $B_1 \in \mathcal{B}$ is a sub-wall of W, then so is $\tilde{B}^* \in \mathcal{B}'$. Therefore, from Theorem 4.2, G' must contain a K_t-minor grasped by W, and it can be found efficiently given $(\mathcal{B}, \mathcal{P})$. □ □

5 Proof of Theorem 2.1

We assume w.l.o.g. that graph G is connected - otherwise it is enough to prove the theorem for the connected component of G containing W. We set $z' = w + 4t$. Let W be the $R \times R$ wall in G. Using Theorem 4.1, we can build a chain $(\mathcal{B}, \mathcal{P})$ of $N = 8D^2(10T+6) + 14T + 8$ basic walls of height at least z' in G, such that each wall $B_i \in \mathcal{B}$ is a sub-wall of W. Let $W' = W'(\mathcal{B}, \mathcal{P})$ be the sub-graph corresponding to the chain of walls. Throughout the proof, we will set $\tau = 2t$, and we will consider the set $\mathcal{S}^*$ of all τ-core walls B'_i with $1 < i < N$, so $|\mathcal{S}^*| \geq 8D^2(10T+6) + 14T + 6$.

If at least one of the core walls $B'_i \in \mathcal{S}^*$ is a type-4 wall, then from Corollary 4.1, we can find a flat sub-wall of W of size $((z'-2\tau) \times (z'-2\tau)) = (w \times w)$. Therefore, we assume from now on that no core wall in $\mathcal{S}^*$ is a type-4 wall.

If the number of type-3 core walls in $\mathcal{S}^*$ is at least $2T$, then from Theorem 4.2, we can efficiently construct a K_t-minor in G grasped by W. If the number of type-1 core walls in $\mathcal{S}^*$ is at least $12T + 6$, then from Theorem 4.3, we can construct a K_t-minor in G grasped by W. Therefore, we assume from now on that the number of type-2 walls in $\mathcal{S}^*$ is at least $8D^2(10T+6)+14T+6-2T-(12T+6) = 8D^2(10T+6)$.

Let $\mathcal{S} \subseteq \mathcal{S}^*$ be the set of all core walls B'_i of type 2, with $1 < i < N$. We select a subset $\mathcal{S}' \subseteq \mathcal{S}$ of $4D^2(10T+6)$ core walls, such that every consecutive pair of such walls is separated by at least one wall. In other words, if we denote $\mathcal{S}' = \left\{B'_{i_1}, B'_{i_2}, \ldots, B'_{i_{4D^2(10T+6)}}\right\}$, where $1 < i_1 < i_2 < \ldots < i_{4D^2(10T+6)} < N$, then for all $1 \leq r < 4D^2(10T+6)$, $i_{r+1} \geq i_r + 2$. Subset $\mathcal{S}'$ can be found using a standard greedy procedure, by ordering the walls in $\mathcal{S}$ in the ascending order of their indices, and then choosing all walls whose location in this ordering is even. We then delete walls from $\mathcal{S}'$ as necessary to ensure that $|\mathcal{S}'| = 4D^2(10T+6)$. We need the following claim, whose proof is omitted from this extended abstract.

Claim 5.1. *There is an efficient algorithm to find a set $\mathcal{P}$ of $10T+6$ disjoint paths in G, such that:*

- *For each path $P \in \mathcal{P}$, its endpoints are labeled x_P and y_P. There is a core wall $B'_{i_P} \in \mathcal{S}'$, such $x_P \in V(B'_{i_P} \setminus \Gamma'_{i_P})$, and $y_P \in V(W' \setminus \mathcal{N}(B_{i_P}))$. Moreover, if $P, P' \in \mathcal{P}$ are distinct, then $i_P \neq i_{P'}$.*
- *The paths in $\mathcal{P}$ are internally disjoint from W'.*

Finally, we show that G contains a graph $H \in \mathcal{H}_3$ as a minor, and then invoke Theorem 3.3. We start with the graph $G' = W' \cup \left(\bigcup_{P\in\mathcal{P}} P\right)$. For each column C_j and row R_i of W', we contract all edges in $C_j \cap R_i$, obtaining a graph H'', which is a subdivision of a grid. We then turn H'' into a grid H', as follows: for each maximal 2-path P of H'' that does not contain the corners of H'', we contract all but one edges of P. So far we have contracted edges of W' to turn it into a grid, but we have made no changes in the paths $P \in \mathcal{P}$. Our last step is to contract, for each path $P \in \mathcal{P}$, all but one edges of P. It is immediate to verify that the resulting graph belongs to the family $\mathcal{H}_3$, and therefore, from Theorem 3.3, it contains a K_t-minor. Let J_1 be the sub-graph of H'

spanned by the first t columns of H'. Since B_1 is a sub-wall of W, J_1 is a contraction of a sub-wall of W, and so from Theorem 3.3, graph G contains a K_t-minor grasped by W, and it can be found efficiently.

6 Proof of Theorem 2.2

We assume w.l.o.g. that graph G is connected - otherwise it is enough to prove the theorem for the connected component of G containing W. We set $z' = w + 4t$. Let W be the $R \times R$ wall in G. Using Theorem 4.1, we can build a chain $(\mathcal{B}, \mathcal{P})$ of $N = 500T + 200$ basic walls of height at least z' in G, such that each wall $B_i \in \mathcal{B}$ is a sub-wall of W. We set $\tau = 2t$, and we will consider the set $\mathcal{S}^*$ of all τ-core walls B'_i with $1 \le i \le N$. Let Γ'_i be the boundary of the τ-core wall B'_i, and let $W' = W'(\mathcal{B}, \mathcal{P})$ be the sub-graph of G corresponding to $(\mathcal{B}, \mathcal{P})$.

For each $B_i \in \mathcal{B}$, we define a pair of vertex subsets $X_i = V(B'_i) \setminus V(\Gamma'_i)$ and $Y_i = V(W') \setminus \mathcal{N}(B_i)$. Notice that $X_i \cap Y_i = \emptyset$ and each pair (x, y) of vertices with $x \in X_i$, $y \in Y_i$ is separated by at least $t + 1$ columns in W'. We denote $M_i = (X_i, Y_i)$, and we call it a *demand pair* for B_i. We say that a path P *routes* the pair M_i, iff one of the endpoints of P belongs to X_i, the other endpoint belongs to Y_i, and P is internally disjoint from W'. Notice that if the endpoints of a path P belong to two distinct sets X_i, X_j, then it is possible that P routes both the pairs M_i, M_j. We say that the pair M_i is *routable* in a sub-graph H of G, iff there is a path P in H that routes M_i. We start with the following theorem, whose proof is almost identical to the proof of Lemma 2.1 in [KTW12] and is omitted.

Theorem 6.1. *There is an efficient algorithm, that returns one of the following: either (1) a set A of at most $40T+20$ vertices of G, and a set $\mathcal{B}' \subseteq \mathcal{B} \setminus \{B_1, B_N\}$ of least $396T+3t+150$ walls, such that for each $B_i \in \mathcal{B}'$, $V(B_i) \cap A = \emptyset$, and M_i is not routable in $G \setminus A$, or (2) a set $\mathcal{P}^*$ of $10T + 6$ disjoint paths in G, such that:*

- *For each path $P \in \mathcal{P}^*$, the endpoints are labeled x_P and y_P, and there is $1 < i_P < N$, such that $x_P \in X_{i_P}$ and $y_P \in Y_{i_P}$;*
- *If $P, P' \in \mathcal{P}^*$ and $P \neq P'$, then $|i_P - i_{P'}| > 1$; and*
- *All paths in $\mathcal{P}^*$ are internally disjoint from W'.*

We apply Theorem 6.1 to our chain of walls $(\mathcal{B}, \mathcal{P})$. Assume first that the outcome of Theorem 6.1 is a set $\mathcal{P}^*$ of $10T + 6$ paths. We show that G contains a graph $H \in \mathcal{H}_3$ as a minor, exactly as in the proof of Theorem 2.1, and then invoke Theorem 3.3. We start with the graph $G' = W' \cup \left(\bigcup_{P \in \mathcal{P}^*} P\right)$. For each column C_j and row R_i of W', we contract all edges in $C_j \cap R_i$, obtaining a graph H'', which is a subdivision of a grid. We then turn H'' into a grid H', as follows: for each maximal 2-path P of H'' that does not contain the corners of H'', we contract all but one edges of P. So far we have contracted edges of W' to turn it into a grid, but we made no changes in the paths $P \in \mathcal{P}^*$. Our last step is to contract, for each path $P \in \mathcal{P}^*$, all but one edges of P. It is easy to verify that the resulting graph belongs to the family $\mathcal{H}_3$. Indeed, Theorem 6.1 ensures that for $P, P' \in \mathcal{P}^*$ where $P \neq P'$, the vertices x_P and $x_{P'}$ belong to core walls $B'_{i_P}, B'_{i_{P'}}$, with $|i_P - i_{P'}| > 1$. In other words, the two walls are separated by at least one wall, and $x_P, x_{P'}$ are separated by at least $2t$ columns. The definition of the pairs M_i ensures that for each path $P \in \mathcal{P}^*$, y_P and x_P are also separated by at least $2t$ columns. For every $P \in \mathcal{P}^*$, $x_P \in X_{i_P} = B'_{i_P} \setminus \Gamma'_{i_P}$, so x_P does not lie in the top $2t$ or the bottom $2t$ rows of the grid. We can now apply Theorem 3.3 to find a K_t-minor grasped by W.

Assume now that the outcome of Theorem 6.1 is a set A of at most $40T + 20$ vertices of G, and a set $\mathcal{B}' \subseteq \mathcal{B} \setminus \{B_1, B_N\}$ of at least $396T + 3t + 150$ walls, such that for each $B_i \in \mathcal{B}'$, $V(B_i) \cap A = \emptyset$, and M_i is not routable in $G \setminus A$. Let $\mathcal{B}'' \subseteq \mathcal{B}'$ be a subset of $132T + t + 50$ walls, such that for each pair $B_i, B_{i'} \in \mathcal{B}''$, with $i \neq i'$, $|i - i'| \ge 3$. We can find $\mathcal{B}''$ by standard methods: order the walls in $\mathcal{B}'$ in their natural left-to-right order, and select all walls whose index is 1 modulo 3 in this ordering, discarding any excess walls as necessary, so $|\mathcal{B}''| = 132T + t + 50$. Lastly, we would like to ensure that $\mathcal{B}''$ does not contain walls B_i with $1 \le i \le t-3$ and $N-t+4 \le i \le N$, by simply removing all such walls from $\mathcal{B}''$. Since there are at most $t-3$ such walls in $\mathcal{B}''$, the final size of $\mathcal{B}''$ is at least $132T + 50$. Notice that if $B_i, B_j \in \mathcal{B}''$ with $i \neq j$, then there is no path P in $G \setminus A$, such that P is internally disjoint from W' and it connects X_i to $V(B_j)$, since $V(B_i) \subseteq Y_i$ and M_i is not routable in $G \setminus A$.

Assume that $A = \{a_1, \ldots, a_m\}$, where $m \le 40T + 20$. Our next step is to gradually construct, for each $1 \le j \le m$, a collection $\mathcal{Q}_j$ of paths, where for each path $Q \in \bigcup_{j=1}^m \mathcal{Q}_j$ there is an index i_Q with $B_{i_Q} \in \mathcal{B}''$, such that Q starts at a vertex of X_{i_Q}, and for $Q \neq Q'$, $i_Q \neq i_{Q'}$. All paths in set $\mathcal{Q}_j$ must terminate at a_j, and all paths in $\bigcup_{j=1}^m \mathcal{Q}_j$ are internally disjoint from $W' \cup A$, and mutually disjoint from each other, except for possibly sharing their last endpoint (we view the paths as directed towards the vertices of A).

We start with $\mathcal{Q}_j = \emptyset$ for all $1 \le j \le m$. We say that vertex $a_j \in A$ is *active* iff $|\mathcal{Q}_j| < 2t$. Let $A^* \subseteq A$ be the set of all vertices that are inactive in the current iteration. We say that a wall $B_i \in \mathcal{B}''$ is *active*, iff no path of $\bigcup_{j=1}^m \mathcal{Q}_j$ starts at a vertex of X_i. An iteration is executed as follows. Assume that there is a path Q in $G \setminus A^*$, connecting a vertex $v \in X_i$, for some active wall B_i, to some active vertex $a_j \in A \setminus A^*$, such that Q contains no vertices of $W' \cup A$ as inner vertices. We claim that Q is disjoint from all paths $Q' \in \bigcup_{j=1}^m \mathcal{Q}_j$, except possibly for sharing the last vertex a_j with Q'. Indeed, assume for contradiction that Q' and Q share some vertex other than a_j, say vertex u. Let $B_{i_{Q'}}$ be the wall to which the first vertex of Q' belongs. Then, since B_i is still active, $i_{Q'} \neq i$ must hold, and so $V(B_{i_{Q'}}) \subseteq Y_i$. Concatenating the segments of Q and Q' between their starting endpoints and u, we obtain a path connecting X_i to Y_i, that does not contain any vertices of A and is internally disjoint from W', contradicting the fact that M_i is not routable in $G \setminus A$. Therefore, Q is disjoint from all paths $Q' \in \bigcup_{j=1}^m \mathcal{Q}_j$, except for possibly sharing its last vertex a_j with Q'. We then add Q to $\mathcal{Q}_j$, and continue to the next iteration. It is easy to see that each iteration can be computed efficiently. The algorithm terminates when we cannot make progress anymore: that is, for each active wall B_i, there is no path Q connecting a vertex in X_i to a vertex in $A \setminus A^*$, such that Q is internally disjoint from $W' \cup A$. It is easy to see that the number of iterations is bounded by $|\mathcal{B}''|$, and so the algorithm can be executed efficiently. Consider the final set A^* of inactive vertices, and the final set $\mathcal{B}^*$ of active walls. We say that Case 1 happens if $|A^*| \geq t-4$; we say that Case 2 happens if $|A^*| \leq t-5$, but $|\mathcal{B}'' \setminus \mathcal{B}^*| \geq 80T + 40 + 6t^2$; otherwise we say that Case 3 happens. We analyze each of the three cases separately.

Case 1. We show that if Case 1 happens, then we can find a model of K_t grasped by W. We define a new graph Z, whose vertex set is $V(Z) = \{v_1, \ldots, v_t, u_1, \ldots, u_{t-4}\}$, and the set of edges is a union of two subsets: $E_1 = \{(v_i, u_j) \mid 1 \leq i \leq t; 1 \leq j \leq t-4\}$, and $E_2 = \{(v_i, v_j) \mid 1 \leq i < j \leq 4\}$. In other words, Z is obtained from $K_{t,t-4}$, by adding the 6 edges connecting all pairs of vertices in $\{v_1, \ldots, v_4\}$. It is easy to see that Z contains a K_t-minor, by contracting, for each $5 \leq i \leq t-4$, the edge (v_i, u_{i+4}). We will show that G contains graph Z as a minor, and provide an efficient algorithm for embedding Z into G. It is then easy to find a model of Z, and consequently, a model of K_t in G.

Since we assume that Case 1 happened, $|A^*| \geq t-4$ when the algorithm terminates. Let $a_1, \ldots, a_{t-4}$ be arbitrary $t-4$ vertices of A^*. We will embed, for each $1 \leq i \leq t-4$, vertex u_i of Z into $\{a_i\}$: the connected sub-graph of G consisting of only the vertex a_i.

We say that a basic wall $B_i \in \mathcal{B}''$ is *bad* iff some vertex of $a_1, \ldots, a_{t-4}$ belongs to $\mathcal{N}(B_i)$. Since every pair of walls in $\mathcal{B}''$ is separated by at least two walls, the number of bad walls in $\mathcal{B}''$ is at most $t-4$.

Consider now some vertex a_j, for $1 \leq j \leq t-4$, and the corresponding set $\mathcal{Q}_j$ of $2t$ paths. We discard from $\mathcal{Q}_j$ all paths that originate at a vertex that belongs to a bad wall. We also discard additional paths from $\mathcal{Q}_j$ as needed, until $|\mathcal{Q}_j| = t$ holds. Let $\mathcal{Q}_j = \left\{Q_1^j, \ldots, Q_t^j\right\}$ be this final set of paths, and for each $1 \leq i \leq t$, we denote by x_i^j the first endpoint of path Q_i^j. We assign the label i to x_i^j, denoting $\ell(x_i^j) = i$. Let $\mathcal{Q} = \bigcup_{j=1}^{t-4} \mathcal{Q}_j$, and let $\tilde{A} = \{a_1, \ldots, a_{t-4}\}$.

Our next step is to define a collection $\mathcal{L} = \{L_1, \ldots, L_t\}$ of t disjoint paths contained in $W' \setminus \tilde{A}$, such that, for each $1 \leq i \leq t$, path L_i contains all vertices whose label is i. We will then embed each vertex v_i of Z into the path L_i. The edge (v_i, u_j) of Z, for $1 \leq i \leq t$, $1 \leq j \leq t-4$, will then be embedded into Q_i^j. Finally, we will define a new set $\mathcal{S}$ of 6 disjoint paths contained in $W' \setminus \tilde{A}$, that connect every pair of paths in $\{L_1, \ldots, L_4\}$. We will ensure that the paths in $\mathcal{S}$ are internally disjoint from the paths in $\mathcal{L}$. They are also guaranteed to be internally disjoint from the paths in $\mathcal{Q}$, since all paths in $\mathcal{S}$ are contained in W'. The paths in $\mathcal{S}$ will be used to embed the edges of E_2.

Recall that the walls $B_1, \ldots, B_{t-3}$ do not belong to $\mathcal{B}''$. Let $B_{i^*} \in \{B_1, \ldots, B_{t-3}\}$ be any of these walls that does not contain vertices of $\tilde{A}$. Similarly, let $B_{i^{**}} \in \{B_{N-t+4,\ldots,B_N}\}$ be any wall that does not contain vertices of $\tilde{A}$. Notice that for each wall $B_i \in \mathcal{B}''$, $i^* < i < i^{**}$ must hold. Let $\tilde{\mathcal{B}} \subseteq \mathcal{B}''$ be the set of all walls B_i that contain the vertices x_q^j, for $1 \leq j \leq t-4$, $1 \leq q \leq t$.

Let $\mathcal{R}$ be any set of t rows of W', such that no vertex in $\tilde{A}$ lies in a row of $\mathcal{R}$. Since there are $4t + w$ rows in W', and $|\tilde{A}| = t-4$, such a set exists. We assume that the paths in $\mathcal{R}$ are ordered in their natural top-to-bottom order.

Consider now some vertex x_q^j, for some $1 \leq j \leq t-4$, $1 \leq q \leq t$, and assume that x_q^j belongs to some basic wall $B_i \in \tilde{\mathcal{B}}$. Recall that the label of x_q^j is q, and $\mathcal{N}(B_i)$ does not contain vertices of $\tilde{A}$. Let S_1 be the set of t vertices lying in the first column of B_{i-1}, that belong to the rows in $\mathcal{R}$, such that exactly one vertex from each row in $\mathcal{R}$ belongs to S_1. Define S_2 similarly for the last column of B_{i+1}. We will construct a set $\mathcal{L}(B_i)$ of t disjoint paths, contained in $W'[\mathcal{N}(B_i)]$, connecting the vertices of S_1 to the vertices of S_2, such that the qth path of $\mathcal{L}(B_i)$ in their natural top-to-bottom order contains x^j_q.

Assume first that x^j_q belongs to some row R_s of B_i. Then $2t < s < z' - 2t + 1$ must hold, as $x^j_q \in X_i$. Let $\mathcal{R}'$ be the set of the top $q-1$ rows of W', the bottom $t-q$ rows of W', and the row R_s (so row R_s is the qth row in the set $\mathcal{R}'$ in their natural top-to-bottom order). Let T_1 be a set of t vertices lying in the last column of B_{i-1}, that belong to the rows of $\mathcal{R}'$, such that exactly one vertex from each row in $\mathcal{R}'$ belongs to T_1. Define T_2 similarly for the first column of B_{i+1}. We now build three sets of paths: $\mathcal{L}^1$ is a set of t disjoint paths contained in B_{i-1}, connecting the vertices of S_1 to the vertices of T_1; $\mathcal{L}^2$ is the set of t disjoint paths containing, for each row $R \in \mathcal{R}'$, the segment of R between the unique vertex of $\mathcal{R}' \cap T_1$ and the unique vertex of $\mathcal{R}' \cap T_2$, and $\mathcal{L}^3$ is a set of t disjoint paths contained in B_{i+1}, connecting the vertices of T_2 to the vertices of S_2 (the existence of the sets $\mathcal{L}^1, \mathcal{L}^3$ of paths follows from Claim 2.2). Let $\mathcal{L}(B_i)$ be the concatenation of the paths in $\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3$. Then the qth path of $\mathcal{L}(B_i)$ in their natural top-to-bottom order must contain x^j_q.

If x^j_q does not belong to a row of B_i, but instead lies on a red path of B_i, let P be that red path, and assume that its two endpoints, u and u' belong to rows R_s and R_{s+1}, respectively. We define sets S_1, T_1 and S_2 exactly as before. We change the definition of the set T_2 slightly: instead of a vertex from row R_s lying in the first column of B_{i+1}, we include a vertex from R_{s+1}, lying in the first column of B_{i+1}. The definitions of the paths $\mathcal{L}^1$ and $\mathcal{L}^3$ remain unchanged. The set $\mathcal{L}^2$ also remains unchanged, except for the path contained in the row R_s. We replace that path with the following path: we include a segment of R_s between with the unique vertex in $T_1 \cap V(R_s)$ and u, the path P, and the segment of R_{s+1} between u' and the unique vertex in $T_2 \cap V(R_{s+1})$. We then let $\mathcal{L}(B_i)$ be the concatenation of $\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3$. Then the qth path of $\mathcal{L}(B_i)$ in their natural top-to-bottom order must contain x^j_q.

For each wall $B_i \in \tilde{\mathcal{B}}$, we have defined a collection $\mathcal{L}(B_i)$ of t disjoint paths, that are contained in $W'[\mathcal{N}(B_i)]$, where for $1 \le j \le t$, the jth path starts and terminates at the jth row of $\mathcal{R}$. We now connect all these paths together, as follows. For each consecutive pair $B_i, B_{i'}$ of walls in $\tilde{\mathcal{B}}$, for each $1 \le j \le t$, let u be the last endpoint of the jth path in $\mathcal{L}(B_i)$, and let u' be the first endpoint of the jth path in $\mathcal{L}(B_{i'})$. Both u and u' must belong to the jth row of $\mathcal{R}$. We use a segment of that row to connect u to u'. Once we process every consecutive pair of walls in $\tilde{\mathcal{B}}$, we obtain a set $\mathcal{L}$ of t disjoint paths, such that, for each $1 \le q \le t$, all vertices whose label is q are contained in the qth path of $\mathcal{R}$. Moreover, all paths in $\mathcal{L}$ are contained in $W' \setminus \tilde{A}$.

Let L_1, L_2, L_3, L_4 be the first four paths of $\mathcal{L}$. We extend the four paths slightly to the right and to the left, as follows. For each $1 \le j \le 4$, let R_{i_j} be the row to which the endpoints of the path L_j belong (that is, R_{i_j} is the jth row of $\mathcal{R}$). Let C_1, C_2, C_3 be the first three columns of B_{i^*}, and let C_4 be the first column of $B_{i^{**}}$. (Recall that $B_{i^*} \in \{B_1, \ldots, B_{t-3}\}$ and $B_{i^{**}} \in \{B_{N-t+4,\ldots,B_N}\}$, and they do not contain vertices of $\tilde{A}$.) We extend L_1 to the left along the row R_{i_1} until it contains a vertex of C_1 (see Figure 3). We extend L_2 to the left along the row R_{i_2} until it contains a vertex of C_3, and we extend it to the right along R_{i_2} until it contains a vertex of C_4. We extend L_3 to the left along the row R_{i_3} until it contains a vertex of C_2. Finally, we extend L_4 to the left along the row R_{i_4} until it contains a vertex of C_1, and we extend L_4 to the right along the row R_{i_4} until it contains a vertex of C_4. This finishes the definition of the paths $L_1, \ldots, L_t$. We embed, for each $1 \le i \le t$, the vertex v_i of graph Z into the path L_i. Each edge (u_j, v_i) of E_1, for $1 \le j \le t-4$, $1 \le i \le t$, is embedded into the path Q^j_i. Since L_i is guaranteed to contain the endpoint x^j_i of Q^j_i (whose label is i), this is a valid embedding. Finally, we need to show how to embed the 6 edges of E_2. Observe that each of the four paths L_1, L_2, L_3, L_4 intersect the column C_3, partitioning it into three segments, each connecting a consecutive pair of these paths. We use these three segments of C_3 to embed the edges (v_1, v_2), (v_2, v_3) and (v_3, v_4). Column C_2 is only intersected by paths L_1, L_3 and L_4. We use the segment of C_2 between rows R_{i_1} and R_{i_3} to embed the edge (v_1, v_3). Column C_1 is only intersected by L_1 and L_4. We use a segment of C_1 between rows R_{i_1} and R_{i_4} to embed the edge (v_1, v_4). Finally, column C_4 is only intersected by L_2 and L_4, and we use it similarly to embed the edge (v_2, v_4).

This finishes the definition of the embedding of the graph Z into G. As observed before, we can now find a model of a K_t-minor in G. It is easy to see that the model of the K_t minor is grasped by W', since there is at least one basic wall B_i, such that every path in $\mathcal{L}$ intersects every column of B_i, and all basic walls B_i are sub-walls of W.

Case 2. If Case 2 happens, then $|\mathcal{B}'' \setminus \mathcal{B}^*| \ge 80T + 40 + 6t^2$. For each vertex $a_j \in A$, for each path $Q \in \mathcal{Q}_j$, let $x(Q)$ denote the endpoint of Q that is different from a_j. We say that a wall $B_i \in \mathcal{B}''$ is *bad* iff $\mathcal{N}(B_i)$ contains

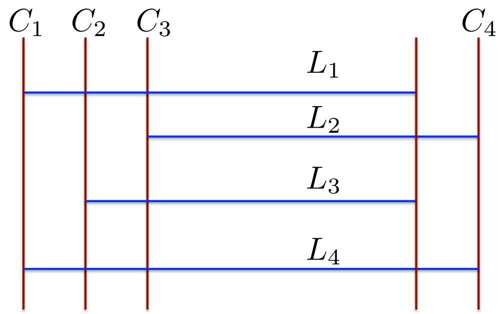

Figure 3: Constructing the paths in $\mathcal{S}$

a vertex of A. Since for each pair $B_i, B_{i'} \in \mathcal{B}''$, with $i \neq i'$, $|i - i'| \geq 3$ holds, the number of bad walls in $\mathcal{B}'' \setminus \mathcal{B}^*$ is at most $|A| \leq 40T + 20$.

For each vertex $a_j \in A$, we delete from $\mathcal{Q}_j$ all paths Q where $x(Q)$ belongs to a bad wall. Then after this procedure, $\sum_{a_j \in A} |\mathcal{Q}_j| \geq |\mathcal{B}'' \setminus \mathcal{B}^*| - (40T + 20) \geq 40T + 20 + 6t^2$, and $\sum_{a_j \in A}(|\mathcal{Q}_j| - 1) \geq 6t^2$.

We discard some additional vertices from A, to obtain the final set $\tilde{A}$. First, for each vertex $a_j \in A$, if $|\mathcal{Q}_j| \leq 1$, then we delete a_j from A. It is easy to see that $\sum_{a_j \in A}(|\mathcal{Q}_j| - 1)$ does not decrease. Finally, let $A' \subset A$ be the subset of vertices that lie in the first $2t$ rows of W', and let $A'' \subset A$ be the subset of vertices lying in the last $2t$ rows of W'. We assume w.l.o.g. that $\sum_{a_j \in A''}(|\mathcal{Q}_j| - 1) \leq \sum_{a_j \in A'}(|\mathcal{Q}_j| - 1)$. We delete the vertices of A'' from A, obtaining the final set $\tilde{A}$. From our construction, $\sum_{a_j \in \tilde{A}}(|\mathcal{Q}_j| - 1) \geq 3t^2$. Let $\mathcal{R}$ be the set of the bottom $2t$ rows of W'. Then no vertex of $\tilde{A}$ belongs to a row of $\mathcal{R}$.

Our next step is to find a collection $A_1, \ldots, A_t$ of t disjoint subsets of $\tilde{A}$, such that for each $1 \leq r \leq t$, $\sum_{a_j \in A_r}(|\mathcal{Q}_j| - 1) \geq t$. We will also delete some paths from sets $\mathcal{Q}_j$ for $a_j \in A_r$, to ensure that this summation is exactly t for each set A_r.

We find the partition of $\tilde{A}$ via a simple greedy procedure. Assume w.l.o.g. that $\tilde{A} = \{a_1, \ldots, a_{m'}\}$. Let j be the smallest index, such that $\sum_{i=1}^{j}(|\mathcal{Q}_i| - 1) \geq t$. Then, since for all $a_i \in \tilde{A}$, $2 \leq |\mathcal{Q}_i| \leq 2t$, $\sum_{i=1}^{j}(|\mathcal{Q}_i| - 1) \leq 3t$. We delete paths from $\mathcal{Q}_j$, until $\sum_{i=1}^{j}(|\mathcal{Q}_i| - 1) = t$ holds. From our choice of j, $|\mathcal{Q}_j| \geq 2$ continues to hold. We set $A_1 = \{a_1, \ldots, a_j\}$, delete the vertices of A_1 from $\tilde{A}$, and continue to the next iteration. Since $\sum_{a_i \in \tilde{A}}(|\mathcal{Q}_i| - 1) \geq 3t^2$, we can continue this process for t iterations, and find the desired collection $A_1, \ldots, A_t$ of subsets of $\tilde{A}$.

Consider some set A_r, for $1 \leq r \leq t$. Recall that for each vertex $a_i \in A_r$, $|\mathcal{Q}_i| \geq 2$. We select one arbitrary path $Q_i^r \in \mathcal{Q}_i$, and we assign to its endpoint $x(Q_i^r)$ the label $(t + r)$. Let $\mathcal{S}_1^r = \{Q_i^r \mid a_i \in A_r\}$ be the set of all paths whose endpoint is assigned the label $(t + r)$, and let $\mathcal{S}_2^r = \left(\bigcup_{a_i \in A_r} \mathcal{Q}_i\right) \setminus \mathcal{S}_1^r$ be the set of the remaining paths. Then $|\mathcal{S}_2^r| = t$. For each $Q \in \mathcal{S}_2^r$, we assign to $x(Q)$ a label in $\{1, \ldots, t\}$, such that each label is assigned to exactly one endpoint $x(Q)$ of a path in $\mathcal{S}_2^r$.

In the rest of the proof, we will embed a $K_{t,t}$-minor into G, as follows. We denote the two sets of vertices in the bi-partition of $K_{t,t}$ by $\{v_1, \ldots, v_t\}$ and $\{u_1, \ldots, u_t\}$. We will build $2t$ paths $P_1, \ldots, P_{2t}$ in $W' \setminus \tilde{A}$, such that for each $1 \leq i \leq 2t$, path P_i contains all vertices with label i. This is done very similarly to the algorithm used in Case 1.

For each $1 \leq r \leq t$, let C_r be the union of the path P_{t+r} and all paths in $\mathcal{S}_1^r$. Note that C_r is a connected graph, as for each $Q \in \mathcal{S}_1^r$, the label of $x(Q)$ is $t + r$, and so P_{t+r} must contain $x(Q)$. For each $1 \leq r \leq t$, we embed the vertex v_r of $K_{t,t}$ into C_r. For each $1 \leq i \leq t$, we embed the vertex u_i of $K_{t,t}$ into the path P_i. The edge (v_r, u_j) is then embedded into the unique path $Q \in \mathcal{S}_2^r$ whose endpoint $x(Q)$ has label j. Then one endpoint of Q belongs to A_r, and hence to C_r, while the other must lie on P_j. This finishes the description of the embedding of $K_{t,t}$ into G. It is now easy to obtain an embedding of K_t into G: for each $1 \leq i \leq t$, we take the union of the embeddings of v_i, u_i and edge (v_i, u_i) to obtain an embedding of the ith vertex of K_t. The edge connecting the ith and the jth vertices of K_t is then embedded into the same path as the edge (v_i, u_j) of $K_{t,t}$. We will ensure that each path P_i that we construct intersects at least t columns of W'. It then follows that the corresponding model of K_t is grasped by W.

It now only remains to define the set $\{P_1, \ldots, P_{2t}\}$ of disjoint paths in $W' \setminus \tilde{A}$, such that for each $1 \leq i \leq 2t$, all vertices whose label is i belong to P_i, and each path P_i intersects at least t columns of W'. Recall that we have defined a set $\mathcal{R}$ of $2t$ rows of W' that do not contain the vertices of $\tilde{A}$. The remainder of the proof closely follows the analysis for Case 1 and is omitted.

Case 3. If Case 3 happens, then $|A^*| \leq t - 5$, and $|\mathcal{B}^*| \geq 132T + 50 - (80T + 40 + 6t^2) \geq 14T + 6$. Recall that we have the following properties for the walls in $\mathcal{B}^*$:

P1. For each $B_i, B_{i'} \in \mathcal{B}^*$ with $i \neq i'$, $|i - i'| \geq 3$;

P2. For each $B_i \in \mathcal{B}^*$, $A \cap V(B_i) = \emptyset$;

P3. For each $B_i \in \mathcal{B}^*$, there is no path Q connecting a vertex of X_i to a vertex of $A \setminus A^*$ in G, such that Q is internally disjoint from $W' \cup A$;

P4. For each $B_i \in \mathcal{B}^*$, there is no path P routing M_i in $G \setminus A^*$.

In order to see that the last property holds, assume for contradiction that there is a path P routing M_i in $G \setminus A^*$. Since from Theorem 6.1 M_i is not routable in $G \setminus A$, path P must contain a vertex of $A \setminus A^*$, contradicting Property (P3).

We say that $B_i \in \mathcal{B}^*$ is a type-$\tilde{1}$ wall iff there is a path P_i connecting X_i to $V(W' \setminus B_i')$ in $G \setminus A^*$, such that P_i is internally disjoint from $W' \cup A^*$. Let $B_i \in \mathcal{B}^*$ be a wall of type-$\tilde{1}$, and let P_i be the corresponding path. We denote by x_i the endpoint of P_i lying in $X_i = V(B_i' \setminus \Gamma_i')$, and by y_i its other endpoint. From Property (P4), $y_i \notin Y_i$, and so $y_i \in \mathcal{N}_i \setminus V(B_i')$ must hold. Observe that if $B_i, B_{i'} \in \mathcal{B}^*$ are type-$\tilde{1}$ walls, then P_i and $P_{i'}$ must be disjoint - otherwise, by combining P_i and $P_{i'}$, we can obtain a path P connecting $x_i \in X_i$ to $x_{i'} \in Y_i$ in graph $G \setminus A^*$, contradicting Property (P4). Let $\mathcal{B}_1^*$ be the set of all type-$\tilde{1}$ walls in $\mathcal{B}^*$.

We say that Case 3a happens if $|\mathcal{B}_1^*| \geq 12T+6$. Assume that Case 3a happens, and consider the sub-graph G' of G, obtained by taking the union of W' and the paths P_i for all $B_i \in \mathcal{B}_1^*$. Then W' is a chain of N walls of height at least z' in G', and for every wall $B_i \in \mathcal{B}_1^*$, the corresponding τ-core wall B_i' is a type-1 wall in G', with P_i being the neighborhood bridge for B_i. Applying Theorem 4.3 to G' and W', we obtain an efficient algorithm to find a model of a K_t-minor in G' (and hence in G), grasped by W.

From now on we assume that Case 3a does not happen. Let $\mathcal{B}_2^* = \mathcal{B}^* \setminus \mathcal{B}_1^*$ and consider any wall $B_i \in \mathcal{B}_2^*$. Then $(G \setminus A^*) \setminus \Gamma_i'$ consists of at least two connected components, with one of them containing $B_i' \setminus \Gamma_i'$. Therefore, there is a separation (X, Y) of $G \setminus A^*$, with $B_i' \subseteq X$, $X \cap Y \subseteq \Gamma_i'$, and for each $B_j \in \mathcal{B}_2^*$ with $j \neq i$, $B_j \subseteq Y$. Recall that the corners a_i', b_i', c_i', d_i' of the wall B_i' are fixed. We assume that they appear on Γ_i' in this order clockwise. If graph X contains a wall-cross for B_i' (that is, a pair of disjoint paths connecting a_i' to c_i' and b_i' to d_i'), then we say that wall B_i is of type $\tilde{2}$. Otherwise, it is of type $\tilde{3}$.

If at least one wall $B_i \in \mathcal{B}_2^*$ is a type-$\tilde{3}$ wall, then from Lemma 4.1, we can efficiently find a flat wall B' of size $((z'-2\tau) \times (z'-2\tau)) = ((z'-4t) \times (z'-4t)) = (w \times w)$ in $G \setminus A^*$, such that B' is a sub-wall of B_i and hence of W.

From now on we assume that all walls in $\mathcal{B}_2^*$ are of type $\tilde{2}$. Recall that $|\mathcal{B}_2^*| \geq 14T + 6 - (12T + 6) \geq 2T$. For each wall $B_i \in \mathcal{B}_2^*$, let Q_i^1, Q_i^2 be the pair of disjoint paths realizing the wall-cross for B_i', and let $\tilde{Q} = \bigcup_{B_i \in \mathcal{B}_2^*} \{Q_i^1, Q_i^2\}$. As in Case 3a, all paths in $\tilde{Q}$ must be completely disjoint, since otherwise we can combine two such paths to obtain a routing of some demand M_i for $B_i \in \mathcal{B}_2^*$ in graph $G \setminus A^*$, contradicting Property (P4). Consider the sub-graph G' of G, obtained by taking the union of W' and the paths in $\tilde{Q}$. Then W' is a chain of N walls of height at least z' in G', and for every wall $B_i \in \mathcal{B}_2^*$, the corresponding τ-core wall B_i' is a type-3 wall in G', with Q_i^1, Q_i^2 being the corresponding wall-cross. Applying Theorem 4.2 to G' and W', we can find a model of a K_t minor in G' (and hence in G), grasped by W.

7 A Lower Bound

In this section we prove Theorem 2.3. We can assume that $w, t \geq 4000$: otherwise, we can use a graph G consisting of a single vertex. We round w down to the closest integral multiple of 4, and we set $w' = w/4 - 8$ and $t' = \lfloor t/30 \rfloor$. In order to construct the graph G, we start with a grid whose height and width is $(w't' - 1)$. For each $0 \leq i < t'$, $0 \leq j < t'$, vertex $v(iw', jw')$ is called a *special vertex*. The unique cell of the grid for which $v(iw', jw')$ is the left top corner is denoted by $Q(iw', jw')$, and we call it a *black cell*. For each black cell $Q(iw', jw')$, we add the two diagonals $(v(iw', jw'), v(iw'+1, jw'+1))$ and $(v(iw'+1, jw'), v(iw', jw'+1))$ to the graph. This completes the definition of the graph G. Clearly, G contains a wall of size $\Omega(w't') = \Omega(wt)$ as a minor. We next prove that G does not contain a K_t-minor.

THEOREM 7.1. *Graph G does not contain a K_t-minor.*

Proof. The proof uses the notions of graph drawing and graph crossing number. A drawing of a graph H in the plane is a mapping, in which every vertex of H is mapped into a point in the plane, and every edge into a continuous curve connecting the images of its endpoints, such that no three curves meet at the same point, and no curve contains an image of any vertex other than its endpoints. A *crossing* in such a drawing is a point where the images of two edges intersect, and the *crossing number* of a graph H, denoted by $\mathsf{cr}(H)$, is the smallest number of crossings achievable by any drawing of H in the plane. We use the following well-known theorem [ACNS82, Lei83].

THEOREM 7.2. *For any graph $G = (V, E)$ with $|E| > 7.5|V|$, $\mathsf{cr}(G) \geq \frac{|E|^3}{33.75|V|^2}$. In particular, for all $n > 16$,* $\mathsf{cr}(K_n) > (n-1)^4/272$.

Assume for contradiction that G contains a K_t-minor, and consider its model f. The main idea is to use the natural drawing ψ of G, that contains $(t'-1)^2$ crossings, together with the model f, to obtain a drawing ψ' of K_t with fewer than $(t-1)^4/272$ crossings, leading to a contradiction. For convenience, instead of defining a drawing of K_t, we define a drawing of another graph H, obtained from K_t by subdividing each edge $e \in E(K_t)$ by two new vertices, $u(e)$ and $u'(e)$. Clearly, a drawing of H with z crossings immediately gives a drawing of K_t with z crossings. We let $V_1 = V(K_t)$, and $V_2 = V(H) \setminus V_1$. For each vertex $v \in V_1$, let $\delta(v)$ be the set of edges of H incident on e. Let E' be the set of edges of H whose both endpoints belong to V_2.

We first define the drawings of the vertices of V_1. For each vertex $v \in V(K_t)$, we select an arbitrary vertex $x_v \in f(v)$. The drawing of v in ψ' is at the same point as the drawing of x_v in ψ.

We now turn to define the drawings of the vertices of V_2 and the edges of H. Along the way, for each vertex $v \in V_1$, we will define a set $\mathcal{P}(v)$ of paths contained in $f(v)$. The paths in $\mathcal{P}(v)$ will be used to define the drawings of the edges in $\delta(v)$. For each edge $e \in E(H)$, we will associate a path $Q(e) \subseteq G$ with e, and we will draw the edge e *along* the drawing of the path $Q(e)$ in ψ. In other words, let γ be the drawing of the path $Q(e)$ in G. The drawing $\psi'(e)$ of e will start at the first endpoint of γ, and then will continue very close to γ, in parallel to it and without crossing it, so that $\psi'(e)$ does not contain the images of any vertices of G, except for the endpoints of γ. It will then terminate at the drawing of the second endpoint of γ. Notice that for now we allow $\psi'(e)$ to self-intersect arbitrarily. Consider now two edges $e, e' \in E(H)$, and their corresponding paths $Q(e), Q(e')$. We distinguish between three types of crossings between $\psi'(e)$ and $\psi'(e')$. Type-1 crossings arise whenever an edge $e^* \in Q(e)$ crosses an edge $e^{**} \in Q(e')$ in ψ. The number of type-1 crossings between the images of e and e' in ψ' is bounded by the number of crossings between the edges of $Q(e)$ and the edges of $Q(e')$ in ψ. We will ensure that for each edge $e^* \in E(G)$, at most $t-1$ paths in $\{Q(e) \mid e \in E(H)\}$ contain e^*. Therefore, the number of type-1 crossings can be bounded by $(t-1)^2$ times the number of crossings in ψ, giving the total bound of $(t-1)^2 \cdot (t'-1)^2$. If two paths $Q(e), Q(e')$ share some edge $e^* \in E(G)$, then the portions of the images of e and e' that are drawn along e^* may cross arbitrarily. Similarly, if $Q(e)$ and $Q(e')$ share some vertex $v \in V(G)$ where v is an inner vertex on both paths, then the images of e and e' may cross arbitrarily next to $\psi(v)$. We call all such crossings type-2 crossings. We also include among type-2 crossings the self-crossings of an image of any edge $e \in E(H)$, that are not type-1 crossings. (We will eventually eliminate all type-2 crossings.) Finally, if an endpoint v of some path $Q(e)$ also belongs to some path $Q(e')$, where $e, e' \in E(H)$ and $e \neq e'$, then we allow the images of e and e' to cross once due to this containment. We call all such crossings *type-3 crossings*. We will ensure that each vertex $v \in V(G)$ may serve as an inner vertex in at most $t-1$ paths $\{Q(e) \mid e \in E(H)\}$, and, since the number of vertices of G serving as endpoints of paths in $\{Q(e) \mid e \in E(H)\}$ is at most $2|E(H)|$, the number of all type-3 crossings will be bounded by $2|E(H)| \cdot (t-1) \leq 6t(t-1)^2$.

We now proceed to define the drawings of the vertices of V_2 and the edges of H, along with the sets $\mathcal{P}(v)$ of paths for all $v \in V_1$. We start with $\mathcal{P}(v) = \emptyset$ for all $v \in V_1$.

Let $e = (v, v')$ be any edge of K_t. Recall that $f(e)$ is an edge e', connecting some vertex $a \in f(v)$ to some vertex $b \in f(v')$. Since $f(v)$ induces a connected sub-graph in G, let P_1 be any simple path connecting x_v to a in $G[f(v)]$. Similarly, let P_2 be any path connecting b to $x_{v'}$ in $G[f(v')]$. Consider the vertices $u(e), u'(e)$ that subdivide the edge e in H, and assume that $u(e)$ lies closer to v than $u'(e)$ in the subdivision. We denote the edges $e_1 = (v, u(e))$, $e_2 = (u(e), u'(e))$, and $e_3 = (u'(e), v')$. We draw the edge e_2 along the drawing $\psi(e')$, where $u(e)$ is drawn at $\psi(a)$, and $u'(e)$ is drawn at $\psi(b)$, and we set $Q(e_2) = (e')$. We draw the edge $e_1 = (v, u(e))$ of H along the path P_1, setting $Q(e_1) = P_1$, and we add P_1 to $\mathcal{P}(v)$. Similarly, we draw the edge $(v', u'(e))$ along the path P_2, setting $Q(e_2) = P_2$. We then add P_2 to $\mathcal{P}(v')$.

Recall that the graphs $\{G[f(v)] \mid v \in V(K_t)\}$ are completely disjoint. Moreover, the edges $\{f(e) \mid e \in E(K_t)\}$ are all distinct, and they do not belong to the graphs $\{G[f(v)] \mid v \in V(K_t)\}$. Therefore, for $v, v' \in V_1$ with $v \neq v'$, the paths in $\mathcal{P}(v)$ and $\mathcal{P}(v')$ are completely disjoint. It is then easy to see that type-2 crossings in ψ' are only possible between the images of edges $e, e' \in E(H)$, where $e, e' \in \delta(v)$ for some $v \in V_1$. Our next step is to re-route the edges of $\delta(v)$ along the paths contained in $f(v)$ in such a way that their corresponding drawings do not have type-2 crossings. In order to do so, we perform a simple un-crossing procedure. Given a pair $e, e' \in \delta(v)$ of edges, whose images have a type-2 crossing in ψ', we remove one of the type-2 crossings, without increasing the total number of crossings in the current drawing, by un-crossing the images of the two edges, as shown in Figure 4. We continue performing this procedure, until for each vertex $v \in V_1$, for every pair $e, e' \in \delta(v)$, the images of e and e' do not have a type-2 crossing. We can still associate, with each edge $e \in \delta(v)$, a path $Q(e) \subseteq G[f(v)]$, such that e is drawn along $Q(e)$. We also eliminate type-2 self-crossings of an edge by simply shortcutting the image of the edge at the crossing point. Eventually, only type-1 and type-3 crossings remain in the graph. An edge $e^* \in E(G)$ may belong to at most $(t-1)$ paths in $\{Q(e) \mid e \in E(H)\}$ (if e lies in $G[f(v)]$ for some $v \in V_1$, then it may only belong to the paths in $\mathcal{P}(v)$; otherwise, it may belong to at most one path $Q(e')$ for $e' \in E'$); similarly, a vertex $u \in V(G)$ may be an inner vertex on at most $(t-1)$ paths in $\{Q(e) \mid e \in E(H)\}$. Therefore, as observed before, the total number of crossings in ψ' is bounded by $(t-1)^2(t'-1)^2+6t(t-1)^2 \le (t-1)^2(\frac{t}{30}-1)^2+6t(t-1)^2 < (t-1)^4/272$, contradicting Theorem 7.2.

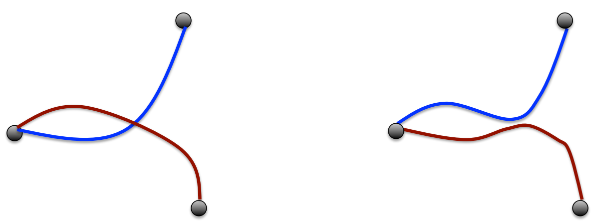

Figure 4: Uncrossing the drawings of a pair of edges to eliminate a type-2 crossing.

$\square$

The following theorem completes the proof of Theorem 2.3.

THEOREM 7.3. *Graph G does not contain a flat wall of size $(w \times w)$.*

Proof. Assume otherwise. Let W be the flat wall of size $(w \times w)$, U the set of the pegs of W, Γ its boundary, and (A, B) the separation of G certifying the flatness of W: that is, $W \subseteq B$, $A \cap B \subseteq \Gamma$, $U \subseteq A \cap B$, and B is $A \cap B$-flat.

For $1 \le i \le w/4$, let W_i be the sub-wall of W spanned by rows $(R_i, \ldots, R_{w-i+1})$ and columns $(C_i, \ldots, C_{w-i+1})$, so $W_1 = W$, and let Γ_i be the boundary of W_i.

Fix some $1 < i \le w/4$. Observe that every path P in graph G connecting a vertex of $W_i \setminus \Gamma_i$ to a vertex of $W \setminus W_i$ must contain a vertex of Γ_i: otherwise, there is a path P' whose endpoints belong to $W_i \setminus \Gamma_i$ and $W \setminus W_i$, respectively, and P' is internally disjoint from W. Path P' must be contained in B, since Γ separates A from B, and P' is internally disjoint from Γ. Then we can use Theorem 2.5 to build a wall-cross for W in graph B, contradicting the fact that W is a flat wall. Therefore, there is a separation (A_i, B_i) of G, with $W_i \subseteq B_i$ and $A_i \cap B_i \subseteq \Gamma_i$, such that $A \subseteq A_i$ and $W \setminus W_i \subseteq A_i$.

Following is the central lemma in the proof of Theorem 7.3.

LEMMA 7.1. *Let $3 \le i \le w/4$, let Q be any black cell, and let S be the set of the 4 vertices serving as the corners of Q. Then $S \not\subseteq B_i$.*

Before we prove Lemma 7.1, let us complete the proof of Theorem 7.3 using it. Let v^* be one of the vertices in the intersection of row $R_{w/2}$ and column $C_{w/2}$ of W. Then there must be a black cell Q in G, such that there is a path P of length at most w' from v^* to one of the corners of the cell Q in G. Let S' be the set of the vertices on P, and the vertices that serve as corners of Q, so $|S'| \le w' + 4$. Consider the cycles $\Gamma_4, \ldots, \Gamma_{w/4}$. Since $|S'| \le w' + 4 < w/4 - 3$, at least one of these cycles Γ_i does not contain any vertex of S'. Therefore, in $G \setminus \Gamma_i$, v^* is connected to all vertices of S', and in particular $S' \subseteq B_i$, a contradiction. From now on we focus on proving Lemma 7.1. Our starting point is the following simple claim, whose proof is omitted.

CLAIM 7.1. *Let $1 \le i \le w/4$, let $Q = Q(i'w', j'w')$ be any black cell, and let S be the set of the four vertices serving as the corners of Q. Assume further that $S \subseteq B_i$. Then there are four disjoint paths in B_i connecting the vertices of S to the vertices of Γ_i.*

We will also repeatedly use the following simple claim, whose proof can be found, e.g. in [RS90].

CLAIM 7.2. *Let H be any graph, X, Y any pair of disjoint vertex subsets of H, and assume that there is a set $\mathcal{P}$ of k disjoint paths connecting the vertices of X to the vertices of Y in H. Let $X' \subseteq X$, and assume that there is a set $\mathcal{P}'$ of $k-1$ disjoint paths connecting the vertices of X' to the vertices of Y in H, such that the paths in $\mathcal{P}'$ are internally disjoint from $X \cup Y$. Then there is a set $\mathcal{P}''$ of k disjoint paths connecting the vertices of X to the vertices of Y in H, such that the paths in $\mathcal{P}''$ are internally disjoint from $X \cup Y$, and $k-1$ of the paths in $\mathcal{P}''$ originate at the vertices of X'.*

We are now ready to complete the proof of Lemma 7.1. Fix some $3 \le i \le w/4$, and let Q be some black cell, such that the set S of the four vertices serving as the corners of Q is contained in B_i. From Claim 7.1, we can find two disjoint paths, P_1, P_2, connecting the vertices of S to the vertices of Γ_i in B_i. Let a, b, c, d be the four corners of the wall W_{i-1}, in this clock-wise order, where a is the top left corner. It is easy to see that we can extend the two paths P_1, P_2, using the edges of $W_i \setminus W_{i-1}$, so that they connect two vertices of S to a and c, the two paths remain disjoint, and are contained in B_{i-1}.

From Claim 7.1, there are three disjoint paths in B_{i-1}, connecting the vertices of S to the vertices of Γ_{i-1}. Using Claim 7.2, we can assume that two of these paths terminate at a and c, respectfully. The third path can then be extended, using the edges of Γ_{i-1}, so that it terminates at either c or d, and it remains disjoint from the first two paths. We assume w.l.o.g. that it terminates at c. Let P'_1, P'_2, P'_3 be the resulting three paths.

Let a', b', c', d' be the four corners W, that appear on Γ in this clock-wise order, where a' is the top left corner. We can extend the three paths P'_1, P'_2, P'_3, using the edges of $E(W) \setminus E(W_{i-1})$, so that they connect three vertices of S to a', b' and c', such that the three paths remain disjoint. It is easy to see that all three paths are contained in $B = B_1$. Finally, using Claim 7.1, there are four disjoint paths in B, connecting the vertices of S to the vertices of Γ. Using Claim 7.2, we can assume that three of these paths terminate at a', b' and c'. The vertices a', b', c' partition Γ into three segments, each of which contains at least two pegs. Let x be the endpoint of the fourth path. Then we can extend the fourth path along Γ, so that it remains disjoint from the first three paths, and it terminates at a peg of W. As the corners of a wall are a subset of its pegs, we now obtained a set $\mathcal{P}$ of four disjoint paths, connecting the vertices of S to the vertices of U, where $\mathcal{P} \subseteq B$. Using the paths in $\mathcal{P}$, and the edges of the cell Q, we can route any matching between the four corresponding pegs in graph B. This contradicts the fact that B is $A \cap B$-flat. This completes the proof of Lemma 7.1, and of Theorem 7.3. $\square$

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