Large Minors in Expanders

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Abstract

In this paper we study expander graphs and their minors. Specifically, we attempt to answer the following question: what is the largest function $f(n,\alpha,d)$, such that every n-vertex α-expander with maximum vertex degree at most d contains **every** graph H with at most $f(n,\alpha,d)$ edges and vertices as a minor? Our main result is that there is some universal constant c, such that $f(n,\alpha,d) \geq \frac{n}{c\log n}\cdot\left(\frac{\alpha}{d}\right)^c$. This bound achieves a tight dependence on n: it is well known that there are bounded-degree n-vertex expanders, that do not contain any grid with $\Omega(n/\log n)$ vertices and edges as a minor. The best previous result showed that $f(n,\alpha,d) \geq \Omega(n/\log^{\kappa} n)$, where κ depends on both α and d. Additionally, we provide a randomized algorithm, that, given an n-vertex α-expander with maximum vertex degree at most d, and another graph H containing at most $\frac{n}{c\log n}\cdot\left(\frac{\alpha}{d}\right)^c$ vertices and edges, with high probability finds a model of H in G, in time $\mathrm{poly}(n)\cdot(d/\alpha)^{O(\log(d/\alpha))}$. We also show a simple randomized algorithm with running time $\mathrm{poly}(n,d/\alpha)$, that obtains a similar result with slightly weaker dependence on n but a better dependence on d and α, namely: if G is an n-vertex α-expander with maximum vertex degree at most d, and H contains at most $\frac{\alpha^3 n}{c' d^5 \log^2 n}$ edges and vertices, where c' is an absolute constant, then our algorithm with high probability finds a model of H in G.

We note that similar but stronger results were independently obtained by Krivelevich and Nenadov: they show that $f(n,\alpha,d) = \Omega(\frac{n\alpha^2}{d^2\log n})$, and provide an efficient algorithm, that, given an n-vertex α-expander of maximum vertex degree at most d, and a graph H with $O(\frac{n\alpha^2}{d^2\log n})$ vertices and edges, finds a model of H in G.

Finally, we observe that expanders are the 'most minor-rich' family of graphs in the following sense: for every n-vertex and m-edge graph G, there exists a graph H with $O(\frac{n+m}{\log n})$ vertices and edges, such that H is not a minor of G.

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1 Introduction

In this paper we study large minors in expander graphs. A graph G is an *α-expander*, if, for every partition (A, B) of its vertices into non-empty subsets, the number of edges connecting vertices of A to vertices of B is at least $\alpha \cdot \min\{|A|, |B|\}$. We say that G is an *expander*, if it is an α-expander for some constant $0 < \alpha < 1$, that is independent of the graph size. A graph H is a *minor* of a given graph G, if one can obtain a graph isomorphic to H from G, via a sequence of edge- and vertex-deletions and edge-contractions.

Bounded-degree expanders are graphs that are simultaneously extremely well connected, while being sparse. Expanders are ubiquitous in discrete mathematics, theoretical computer science and beyond, arising in a wide variety of fields ranging from computational complexity to designing robust computer networks (see [HLW06] for a survey on expanders and their applications). In this paper we study an extremal problem about expanders: what if the largest function $f(n, \alpha, d)$, such that every n-vertex α-expander with maximum vertex degree at most d contains *every* graph with at most $f(n, \alpha, d)$ vertices and edges as a minor?

Our main result is that there is a constant c, such that $f(n, \alpha, d) \geq \frac{n}{c \log n} \cdot \left(\frac{\alpha}{d}\right)^c$. As we discuss below, this result achieves an optimal dependence on n. We also provide a randomized algorithm that, given an n-vertex α-expander with maximum vertex degree at most d, and another graph H containing at most $\frac{n}{c \log n} \cdot \left(\frac{\alpha}{d}\right)^c$ edges and vertices, with high probability finds a model of H in G, in time $\text{poly}(n) \cdot (d/\alpha)^{\log(d/\alpha)}$. Additionally, we show a simple randomized algorithm with running time $\text{poly}(n, d/\alpha)$, that achieves a bound that has a slightly worse dependence on n but a better dependence on d and α: if G is an n-vertex α-expander with maximum vertex degree at most d, and H is any graph with at most $\frac{\alpha^3 n}{c' d^5 \log^2 n}$ edges and vertices, for some universal constant c', the algorithm finds a model of H in G with high probability.

Independently from our work, Krivelevich and Nenadov (see Theorem 8.1 in [Kri18a]) provide an elegant proof of a similar but stronger result: namely, they show that $f(n, \alpha, d) = \Omega(\frac{n\alpha^2}{d^2 \log n})$, and provide an efficient algorithm, that, given an n-vertex α-expander of maximum vertex degree at most d, and a graph H with $O(\frac{n\alpha^2}{d^2 \log n})$ vertices and edges, finds a model of H in G.

One of our main motivations for studying this question is the Excluded Grid Theorem of Robertson and Seymour. This is a fundamental result in graph theory, that was proved by Robertson and Seymour [RS86] as part of their Graph Minors series. The theorem states that there is a function $t : \mathbb{Z}^+ \to \mathbb{Z}^+$, such that for every integer $g > 0$, every graph of treewidth at least $t(g)$ contains the $(g \times g)$-grid as a minor. The theorem has found many applications in graph theory and algorithms, including routing problems [RS95], fixed-parameter tractability [DH08, DH07], and Erdős-Pósa-type results [RS86, Car88, Ree97, FST11]. For an integer $g > 0$, let $t(g)$ be the smallest value, such that every graph of treewidth at least $t(g)$ contains the $(g \times g)$-grid as a minor. An important open question is establishing tight bounds on the function t. Besides being a fundamental graph-theoretic question in its own right, improved upper bounds on t directly affect the running times of numerous algorithms that rely on the theorem, as well as parameters in various graph-theoretic results, such as, for example, Erdős-Pósa-type results.

In a series of works [RS86, RST94, KK12, LS15, CC16, Chu15, Chu16a, CT], it was shown that $t(g) = \tilde{O}(g^9)$ holds. The best currently known negative result, due to Robertson et al. [RST94] is that $t(g) = \Omega(g^2 \log g)$. This is shown by employing a family bounded-degree expander graphs of large girth. Specifically, consider an n-vertex expander G whose maximum vertex degree is bounded by a constant independent of n, and whose girth is $\Omega(\log n)$. It is not hard to show that the treewidth of G is $\Omega(n)$. Assume now that G contains the $(g \times g)$-grid as a minor, for some value g. Such a grid contains $\Omega(g^2)$

disjoint cycles, each of which must consume $\Omega(\log n)$ vertices of G, and so $g \le O(\sqrt{n/\log n})$. This simple argument is the best negative result that is currently known for the Excluded Grid Theorem. In fact, Robertson and Seymour conjecture that this bound is tight, that is, $t(g) = \Theta(g^2 \log g)$ must hold. A natural question therefore is whether this analysis is tight, and in particular, whether every n-vertex bounded-degree expander must contain a $(g \times g)$-grid as a minor, for $g = O(\sqrt{n/\log n})$. In this paper we answer this question in the affirmative, and moreover, we show that *every* graph with at most $O(n/\log n)$ vertices and edges is a minor of such an expander.

The problem of finding large minors in bounded-degree expanders was first considered by Kleinberg and Rubinfield [KR96]. Building on the random walk-based techniques of Broder et al. [BFU94], they showed that every expander G on n vertices contains every graph with $O(n/\log^{\kappa} n)$ vertices and edges as a minor. The exponent κ depends on the expansion α and the maximum degree d of the expander; we estimate it to be at least $\Theta(\log^2 d/\log^2(1/\alpha))$. They also show an efficient algorithm for finding a model of such a graph in G.

Another related direction of research is the existence of large clique minors in graphs. The study of the size of the largest clique minor in a graph is motivated by Hadwiger's conjecture, which states that, if the chromatic number of a graph is at least k, then it contains a clique with k vertices as a minor. One well-known result in this area, due to Kawarbayashi and Reed [KR10], shows that every bounded-degree expander G with n vertices contains a clique with $\Omega(\sqrt{n})$ vertices as a minor. Notice that this is a tight bound, since G contains only $O(n)$ edges. Our results imply a slightly weaker bound of $\Omega(\sqrt{n/\log n})$ on the size of the clique minor.

The existence of large clique minors was also studied in the context of random graphs. Recall that $G \sim \mathcal{G}(n,p)$ is a random graph on n vertices, whose edges are added independently with probability p each. Bollobás, Catlin and Erdős [BCE80] showed that Hadwiger's conjecture is true for almost all graphs $\mathcal{G}(n,p)$ for every constant $p > 0$. Fountoulakis et al. [FKO09] later showed that for every $\epsilon > 0$, there is a constant $\hat{c}_\epsilon$ such that the following is true: if $q(n,\epsilon)$ is the probability that the graph $G \sim \mathcal{G}(n, \frac{1+\epsilon}{n})$ does not contain a clique minor on $\lceil \hat{c}_\epsilon \sqrt{n} \rceil$ vertices, then $\lim_{n\to\infty} q(n,\epsilon) = 0$. Using a theorem from [Kri18b], our results imply a slightly weaker bound of $\Omega(\sqrt{n/\log n})$ on the clique minor size.

Our Results and Techniques. All graphs that we consider are finite; they do not have loops or parallel edges. Given a graph H, we define its *size* to be $|V(H)| + |E(H)|$. Our main result is summarized in the following theorem:

Theorem 1.1 *There is a constant c^*, such that for all $0 < \alpha < 1$ and $d \ge 1$, if G is an n-vertex α-expander with maximum vertex degree at most d, and H is any graph of size at most $\frac{n}{c^* \log n} \cdot \left(\frac{\alpha}{d}\right)^{c^*}$, then H is a minor of G. Moreover, there is a randomized algorithm, whose running time is* $\operatorname{poly}(n) \cdot (d/\alpha)^{O(\log(d/\alpha))}$*, that, given G and H as above, with high probability, finds a model of H in G.*

As discussed above, the theorem implies that we cannot get stronger negative results for the Excluded Grid Theorem using bounded-degree α-expanders, where α is independent of the graph size. But this leaves open the possibility of obtaining stronger negative results when α is a function of n, such as, for example, $\alpha = 1/\operatorname{poly}\log n$, or $\alpha = 1/n^{\epsilon}$ for some small constant ϵ. Our next result provides a simpler algorithm, with better running time and a better dependence on d and α, at the cost of slightly weaker dependence on n in the minor size.

Theorem 1.2 *There is a constant $\tilde{c}^*$ and a randomized algorithm, that, given an n-vertex α-expander with maximum vertex degree at most d, where $0 < \alpha < 1$, and another graph H of size at most $\frac{\alpha^3 n}{\tilde{c}^* d^5 \log^2 n}$, with high probability computes a model of H in G, in time* $\operatorname{poly}(n, d/\alpha)$.

The following corollary easily follows from Theorem 1.1 and a result of [Kri18b].

Corollary 1.3 *For every $\epsilon > 0$, there is a constant c_ϵ depending only on ϵ, such that a random graph $G \sim \mathcal{G}(n, \frac{1+\epsilon}{n})$ with high probability contains every graph of size at most $c_\epsilon n/\log n$ as a minor.*

As mentioned earlier, similar but somewhat stronger results were obtained independently by Krivelevich and Nenadov (see Theorem 8.1 in [Kri18a]).

As a final comment, we show in Appendix B that expanders are the 'most minor-rich' family of graphs:

Observation 1.4 *For every graph G of size $s \geq 2$, there is a graph H_G of size at most $20s/\log s$ such that G does not contain H_G as a minor.*

We now turn to describe our techniques, starting with the simpler result: Theorem 1.2. Given an n-vertex α-expander G with maximum vertex degree at most d, we compute a partition of G into two disjoint subgraphs, G_1 and G_2, such that G_1 is a connected graph; G_2 is an α'-expander for a somewhat weaker parameter α', and a large matching $\mathcal{M}$ connecting vertices of G_1 to vertices of G_2. We refer to the edges of $\mathcal{M}$, and to their endpoints, as *terminals*. Assume now that we are given a graph H, containing at most $\frac{\alpha^3 n}{\tilde{c}^* d^5 \log^2 n}$ vertices and edges. We can assume w.l.o.g. that the maximum vertex degree in H is at most 3, as we can compute a graph H' of size at most twice the size of H, such that the maximum vertex degree of H' is at most 3, and H is a minor of H'. Using the transitivity of the minor relation, it is now sufficient to show that H' is a minor of G. Therefore, we assume that the maximum vertex degree in H is at most 3, and we denote $|V(H)| = n'$. Using the standard grouping technique, we partition the graph G_1 into connected subgraphs $S_1, \ldots, S_{n'}$, each of which contains at least $\Theta(d^2 \log^2 n/\alpha^2)$ terminals. Assume that $H = \{v_1, \ldots, v_{n'}\}$. We map the vertex v_i of H to the graph S_i. Let $E_i \subseteq \mathcal{M}$ be the set of edges of $\mathcal{M}$ incident to the vertices of S_i. Every edge $(v_i, v_j) \in E(H)$ is embedded into a path in the expander G_2, that connects some edge of E_i to some edge of E_j. The paths are found using standard techniques: we use the classical result of Leighton and Rao [LR99] to show that for every edge $e = (v_i, v_j)$ of H, there is a large set $\mathcal{P}_e$ of paths in G_2, connecting edges of E_i to edges of E_j, such that all resulting paths in $\mathcal{P} = \bigcup_{e \in E(H)} \mathcal{P}_e$ are short, and cause a small vertex-congestion in G_2. We then use the constructive proof of the Lovasz Local Lemma by Moser and Tardos [MT10] to select a single path P_e from each such set $\mathcal{P}_e$, so that the resulting paths are disjoint in their vertices.

The proof of Theorem 1.1 is somewhat more complex. As before, we assume w.l.o.g. that maximum vertex degree in the graph H is at most 3. We define a new combinatorial object called a Path-of-Expanders System (see Figure 1). At a high level, a Path-of-Expanders System of width w and expansion α' consists of 12 graphs: graphs $T_1, \ldots, T_6$ that are α'-expanders, and graphs $S_1, \ldots, S_6$ that are connected graphs. For each $1 \leq i \leq 6$, we are also given a matching $\mathcal{M}'_i$ of cardinality w connecting vertices of S_i to vertices of T_i; the endpoints of the edges of $\mathcal{M}'_i$ in S_i and T_i are denoted by B_i and C_i, respectively. For each $1 \leq i < 6$, we are given a matching $\mathcal{M}_i$ connecting every vertex of B_i to some vertex of S_{i+1}; the endpoints of the edges of $\mathcal{M}_i$ that lie in S_{i+1} are denoted by A_{i+1}. We show that an n-vertex α-expander with maximum vertex degree at most d must contain a Path-of-Expanders System of width $w \geq n(\alpha/d)^c$ and expansion $\alpha' = (\alpha/d)^{c'}$ for some constants c and c', and provide an algorithm with running time $\text{poly}(n) \cdot (d/\alpha)^{O(\log(d/\alpha))}$ to compute it. Next, we split the Path-of-Expanders System into three parts. The first part is the union of the graphs S_2, T_2 and the matching $\mathcal{M}'_2$. We view the vertices of B_2 as terminals, and we use the graph T_2 and the matching $\mathcal{M}'_2$ in order to partition them into large enough groups, and to define a connected sub-graph of $T_2 \cup \mathcal{M}'_2$ spanning each such group, like in the proof of Theorem 1.2. We ensure that the number of groups is equal to the number of vertices in the graph H that we are trying to embed into G. Every vertex of

H is then embedded into a separate group, together with the corresponding connected sub-graph of $T_2 \cup \mathcal{M}'_2$ spanning the group.

We use the graphs $S_3, \ldots, S_6, T_3 \ldots, T_6$ in order to route all but a small fraction of the edges of H. The algorithm in this part is inspired by the algorithm of Frieze [Fri01] for routing a large set of demand pairs in an expander graph via edge-disjoint paths. Lastly, the remaining edges of H are routed in graph $S_1 \cup T_1 \cup \mathcal{M}'_1$, using essentially the same algorithm as the one in the proof of Theorem 1.2.

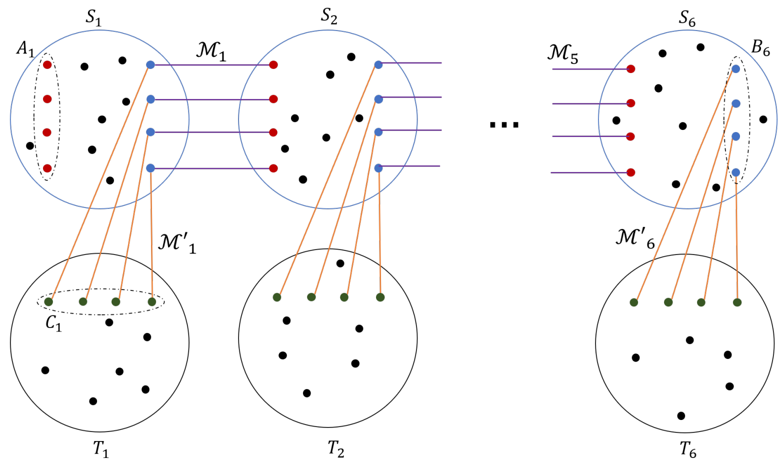


Figure 1: An illustration of the Path-of-Expanders System $\Pi = (\mathcal{S}, \mathcal{M}, A_1, B_6, \mathcal{T}, \mathcal{M}')$. For each $1 \le i \le 6$, the vertices of A_i, B_i and C_i are shown in red, blue and green, respectively.

Organization. We start with Preliminaries in Section 2. The proof of Theorem 1.1 is provided in Section 3, with some of the technical details deferred to Sections 4 and 5. Section 6 contains an algorithm for constructing a Path-of-Expanders System. The proof of Theorem 1.2 appears in Section 7, and the proofs of Corollary 1.3 and Observation 1.4 appear in Sections A and B of the Appendix, respectively.

2 Preliminaries

Throughout the paper, for an integer $\ell \ge 1$, we denote $[\ell] = \{1, \ldots, \ell\}$. All logarithms in the paper are to the base of 2.

All graphs that we consider are finite; they do not have loops or parallel edges.

We will use the following simple observation, whose proof is deferred to the Appendix.

Observation 2.1 *There is an efficient algorithm, that, given a set $\{x_1, \ldots, x_r\}$ of non-negative integers, with $\sum_i x_i = N$, and $x_i \le 3N/4$ for all i, computes a partition (A, B) of $\{1, \ldots, r\}$, such that $\sum_{i \in A} x_i \ge N/4$ and $\sum_{i \in B} x_i \ge N/4$.*

Given a graph $G = (V, E)$ and a subset $V' \subseteq V$ of its vertices, we denote by $\delta_G(V')$ the set of all edges that have exactly one endpoint in V', and by $E_G[V']$ the set of all edges with both endpoints in V'. For readability, we write $\delta_G(v)$ instead of $\delta_G(\{v\})$. Given a pair $V', V'' \subseteq V$ of disjoint subsets of vertices, we denote by $E_G(V', V'')$ the set of all the edges with one endpoint in V' and another in V''. We will omit the subscript G when the underlying graph is clear from context. For a subset $V' \subseteq V$ of vertices of G, we denote by $G[V']$ the subgraph of G induced by V'.

Given a path P in a graph G, we denote by V_P and E_P the sets of all its vertices and edges, respectively. Given a path P and a set V' of vertices of G, we say that P is *disjoint* from V' iff $V_P \cap V' = \emptyset$. We say that P is *internally disjoint* from V' iff every vertex of $V' \cap V_P$ is an endpoint of P.

Similarly, suppose we are given two paths P, P' in a graph G. We say that the two paths are *disjoint* iff $V_P \cap V_{P'} = \emptyset$, and we say that they are *internally disjoint* iff all vertices in $V_P \cap V_{P'}$ serve as endpoints of both these paths.

Let $\mathcal{P}$ be any set of paths in a graph G. We say that $\mathcal{P}$ is a set of *disjoint* paths iff every pair $P, P' \in \mathcal{P}$ of distinct paths are disjoint. We say that $\mathcal{P}$ is a set of *internally disjoint* paths iff every pair $P, P' \in \mathcal{P}$ of distinct paths are internally disjoint. We denote by $V(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} V_P$ the set of all vertices participating in the paths of $\mathcal{P}$. Given a pair V', V'' of subsets of vertices of V (that are not necessarily disjoint), we say that a path $P \in \mathcal{P}$ *connects* V' to V'' iff one of its endpoints is in V' and the other endpoint is in V''. We use a shorthand $\mathcal{P} : V' \rightsquigarrow V''$ to indicate that $\mathcal{P}$ is a collection of disjoint paths, where each path $P \in \mathcal{P}$ connects V' to V''. Notice that each path in $\mathcal{P}$ must originate at a distinct vertex of V' and terminate at a distinct vertex of V''.

Finally, assume that we are given a (partial) matching $\mathcal{M}$ over the vertices of G, and a set $\mathcal{P}$ of $|\mathcal{M}|$ paths. We say that $\mathcal{P}$ *routes* $\mathcal{M}$ iff for every pair of vertices $(v', v'') \in \mathcal{M}$, there is a path $P \in \mathcal{P}$, whose endpoints are v' and v''.

Sparsest Cut and Expansion. A *cut* in G is a bipartition (S, S') of its vertices, that is, $S \cup S' = V$, $S \cap S' = \emptyset$ and $S, S' \neq \emptyset$. The *sparsity* of the cut (S, S') is $|E(S, S')| / \min\{|S|, |S'|\}$. The *expansion* of a graph G, denoted by $\varphi(G)$, is the minimum sparsity of any cut in G.

Definition 1 *Given a parameter $\alpha > 0$, we say that a graph G is an α-expander iff $\varphi(G) \geq \alpha$. Equivalently, for every subset S of at most $|V(G)|/2$ vertices of G, $|\delta_G(S)| \geq \alpha |S|$.*

The following theorem follows from the standard Cheeger's inequality, that shows that for any graph G, whose maximum vertex degree is bounded by d, $\frac{\lambda(G)}{2} \leq \varphi(G) \leq \sqrt{2d\lambda(G)}$, where $\lambda(G)$ is the second smallest eigenvalue of the Laplacian of G, and from the algorithm of [Fie73] (see also [AM84, Alo86, Alo98]).

Theorem 2.2 *There is an efficient algorithm, that, given an n-vertex graph G with maximum vertex degree at most d, computes a cut (A, B) in G of sparsity $O(\sqrt{d\varphi(G)})$.*

Finally, we use the following simple claim several times; the claim allows one to "fix" an expander, after a small number of edges were deleted from it. The proof appears in Appendix.

Claim 2.3 *Let T be an α-expander, and let E' be any subset of edges of T. Then there is an $\alpha/4$-expander $T' \subseteq T \setminus E'$, with $|V(T')| \geq |V(T)| - \frac{4|E'|}{\alpha}$.*

Graph Minors.

Definition 2 (Graph Minors) *We say that a graph $H = (U, F)$ is a* minor *of a graph $G = (V, E)$ iff there is a map f, called a* model of H in G, *mapping every vertex $u \in U$ to a subset $X_u \subseteq V$ of vertices, and mapping every edge $e \in F$ to a path P_e in G, such that:*

- *For every vertex $u \in U$, $G[X_u]$ is connected;*
- *For every edge $e = (u, v) \in F$, the path P_e connects X_u to X_v;*

- *For every pair $u, v \in U$ of distinct vertices, $X_u \cap X_v = \emptyset$; and*
- *Paths $\{P_e \mid e \in F\}$ are internally disjoint from each other and they are internally disjoint from the set $\bigcup_{u \in U} X_u$ of vertices.*

For a vertex $u \in U$ we sometimes call $G[X_u]$ the embedding of u into G*, and for an edge $e \in F$, we sometimes refer to P_e as the* embedding of e into G*.*

Well-Linkedness and Path-of-Sets System. We use a slight variation of the standard definition of (node)-well-linkedness.

Definition 3 (Well-Linkedness) *We say that a set A of vertices in a graph G is* well-linked *iff for every pair A', A'' of disjoint equal-cardinality subsets of A, there is a set $\mathcal{P} : A' \rightsquigarrow A''$ of $|A'|$ paths in G, that are internally disjoint from A. (Note that the paths in $\mathcal{P}$ must be disjoint).*

Next, we define a Path-of-Sets system, that was first introduced in [CC16] (a somewhat similar object called *grill* was introduced by [LS15]), and was used since then in a number of graph theoretic results.

Definition 4 (Path-of-Sets System) *Given integers $w, \ell > 0$ a Path-of-Sets System of width w and length ℓ (see Figure 2) consists of:*

- *a sequence $\mathcal{S} = (S_1, \ldots, S_\ell)$ of ℓ disjoint connected graphs, that we refer to as* clusters*;*
- *for each $1 \le i \le \ell$, two disjoint subsets, $A_i, B_i \subseteq V(S_i)$ of w vertices each; and*
- *For each $1 \le i < \ell$, a collection $\mathcal{M}_i$ of edges, connecting every vertex of B_i to a distinct vertex of A_{i+1}.*

We denote the Path-of-Sets System by $\Sigma = (\mathcal{S}, \mathcal{M}, A_1, B_\ell)$, where $\mathcal{M} = \bigcup_i \mathcal{M}_i$. We also denote by G_Σ the graph defined by the Path-of-Sets System, that is, $G_\Sigma = \left(\bigcup_{i=1}^{\ell} S_i\right) \cup \mathcal{M}$.

We say that a given Path-of-Sets System is a Strong Path-of-Sets System *iff all $1 \le i \le \ell$, the vertices of $A_i \cup B_i$ are well-linked in S_i. We say that it is α-expanding, iff for all $1 \le i \le \ell$, graph S_i is an α-expander. Note that a Strong Path-of-Sets System is not necessarily α-expanding and vice versa.*

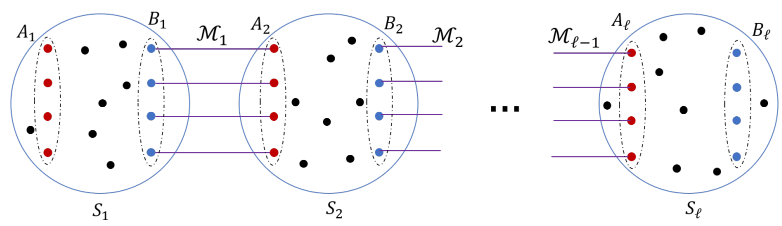


Figure 2: An illustration of a Path-of-Sets System $(\mathcal{S}, \mathcal{M}, A_1, B_\ell)$. For each $i \in [\ell]$, the vertices of A_i and B_i are shown in red and blue respectively.

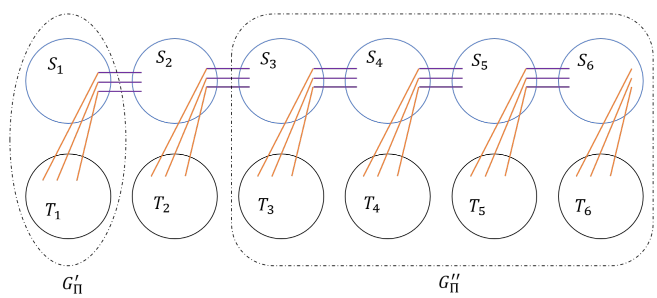


Figure 3: An illustration of the subgraphs G'_Π and G''_Π of G_Π.

2.1 Path-of-Expanders System

Path-of-Expanders System is the main new structural object that we use.

Definition 5 (Path-of-Expanders System) *Given an integer $w > 0$ and a parameter $0 < \alpha < 1$, a Path-of-Expanders System of width w and expansion α (see Figure 1) consists of:*

- *a Strong Path-of-Sets System $\Sigma = (\mathcal{S}, \mathcal{M}, A_1, B_6)$ of width w and length 6;*
- *a sequence $\mathcal{T} = (T_1, \ldots, T_6)$ of 6 disjoint connected graphs, such that for each $1 \leq i \leq 6$, T_i is disjoint from $S_1, \ldots, S_6$, and it is an α-expander; and*
- *for each $1 \leq i \leq 6$, a perfect matching $\mathcal{M}'_i$ between B_i and some subset C_i of w vertices of T_i.*

We denote the Path-of-Expanders System by $\Pi = (\mathcal{S}, \mathcal{M}, A_1, B_6, \mathcal{T}, \mathcal{M}')$, where $\mathcal{M}' = \bigcup_i \mathcal{M}'_i$. For convenience, for each $1 \leq i \leq 6$, we denote by W_i be the graph obtained from the union of the graphs S_i and T_i, and the matching $\mathcal{M}'_i$.

Similarly to the Path-of-Sets System, we associate with the Path-of-Expanders System Π a graph G_Π, obtained by taking the union of the graphs $S_1, \ldots, S_6$, $T_1, \ldots, T_6$ and the sets $\mathcal{M}, \mathcal{M}'$ of edges.

We will be interested in three subgraphs of G_Π (see Figure 3): (i) Graph W_1, that we denote by G'_Π; (ii) Graph W_2; and (iii) Graph G''_Π, obtained by taking the union of $W_3 \cup W_4 \cup W_5 \cup W_6$ and the edges of $\mathcal{M}_3 \cup \mathcal{M}_4 \cup \mathcal{M}_5$.

Definition 6 *We say that a graph G contains a Path-of-Sets System of width w and length ℓ as a minor iff there is a Path-of-Sets System Σ of width w and length ℓ, such that its corresponding graph G_Σ is a minor of G. Similarly, we say that a graph G contains a Path-of-Expanders System of width w and expansion α as a minor iff there is a Path-of-Expanders System Π of width w and expansion α, such that its corresponding graph G_Π is a minor of G.*

The following theorem, that we prove in Section 6, shows that an expander must contain a Path-of-Expanders System with suitably chosen parameters, and provides an algorithm to compute its model in the expander.

Theorem 2.4 *There are constants $\hat{c}_1, \hat{c}_2$, and an algorithm, that, given an α-expander G with $|V(G)| = n$, whose maximum vertex degree is at most d, and $0 < \alpha < 1$, constructs a Path-of-Expanders System Π of expansion $\tilde{\alpha} \geq \left(\frac{\alpha}{d}\right)^{\hat{c}_1}$ and width $w \geq n \cdot \left(\frac{\alpha}{d}\right)^{\hat{c}_2}$, such that the corresponding graph G_Π has maximum vertex degree at most $d+1$ and is a minor of G. Moreover, the algorithm computes a model of G_Π in G. The running time of the algorithm is* $\operatorname{poly}(n) \cdot (d/\alpha)^{O(\log(d/\alpha))}$.

3 Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1. We prove it by using the following theorem.

Theorem 3.1 *There is constant c_0 and a randomized algorithm, that, given a Path-of-Expanders System Π with expansion α and width w, such that the maximum vertex degree in G_Π is at most d and $|V(G_\Pi)| \leq n$ for some $n > c_0$, together with a graph H of maximum vertex degree at most 3 and $|V(H)| \leq \frac{w^2\alpha^2}{2^{19}d^4 n \log n}$, with high probability, in time* $\operatorname{poly}(n)$, *finds a model of H in G_Π.*

Before proving Theorem 3.1, we complete the proof of Theorem 1.1 using it. Let G be the given α-expander with $|V(G)| = n$, and maximum vertex degree at most d. Recall that $0 < \alpha < 1$. By letting c^* be a sufficiently large constant, we can assume that n is sufficiently large, so that, for example, $n > c_0$, where c_0 is the constant from Theorem 3.1. Indeed, otherwise, it is enough to show that the graph with 1 vertex is a minor of G, which is trivially true. Therefore, we assume from now on that n is sufficiently large. From Theorem 2.4, G contains as a minor a Path-of-Expanders System Π of width $w \geq n \cdot \left(\frac{\alpha}{d}\right)^{\hat{c}_2}$ and expansion $\tilde{\alpha} \geq \left(\frac{\alpha}{d}\right)^{\hat{c}_1}$, such that the maximum vertex degree in G_Π is at most $d+1$. Using these bounds, we get that:

$$\begin{aligned}\frac{w^2\tilde{\alpha}^2}{2^{19}(d+1)^4 n \log n} &\geq n^2 \cdot \left(\frac{\alpha}{d}\right)^{2\hat{c}_2} \cdot \left(\frac{\alpha}{d}\right)^{2\hat{c}_1} \cdot \frac{1}{2^{23}d^4 n \log n} \\ &= \frac{n\alpha^{2(\hat{c}_1+\hat{c}_2)}}{2^{23}d^{4+2(\hat{c}_1+\hat{c}_2)} \log n} \\ &\geq \frac{3n}{c^* \log n} \cdot \left(\frac{\alpha}{d}\right)^{c^*},\end{aligned}$$

for $c^* \geq \max\left\{4 + 2(\hat{c}_1 + \hat{c}_2), c_0, 2^{25}\right\}$. Therefore, if H' is a graph with maximum vertex degree at most 3, and $|V(H')| \leq \frac{3n}{c^* \log n} \cdot \left(\frac{\alpha}{d}\right)^{c^*}$, then, from Theorem 3.1, G contains H' as a minor, and from Theorems 2.4 and 3.1, its model in G can be computed with high probability by a randomized algorithm, in time $\operatorname{poly}(n) \cdot (d/\alpha)^{O(\log(d/\alpha))}$.

Consider now any graph $H = (U, F)$ of size at most $\frac{n}{c^* \log n} \cdot \left(\frac{\alpha}{\Delta}\right)^{c^*}$. Let $n' = |U|$ and $m' = |F|$, so $n' + m' \leq \frac{n}{c^* \log n} \cdot \left(\frac{\alpha}{d}\right)^{c^*}$. We construct another graph H', whose maximum vertex degree is at most 3 and $|V(H')| \le n' + 2m'$, such that H is a minor of H'. Since H' must be a minor of G, it follows that H is a minor of G. In order to construct graph H' from H, we consider every vertex $u \in U$ of degree $d_u > 3$ in turn, and replace it with a cycle C_u on d_u vertices, such that every edge incident to u in H is incident to a distinct vertex of C_u. It is easy to verify that the resulting graph H' has maximum vertex degree at most 3, that H is a minor of H', and that $|V(H')| \le 2m' + n'$, completing the proof of Theorem 1.1. Notice that this proof is constructive, that is, there is a randomized algorithm that constructs a model of H in G in time $\text{poly}(n) \cdot (d/\alpha)^{O(\log(d/\alpha))}$. The remainder of this section is dedicated to proving Theorem 3.1, with some details deferred to subsequent sections.

3.1 Large Minors in Path-of-Expanders System

This subsection is devoted to the proof Theorem 3.1. We assume that we are given a Path-of-Expanders System $\Pi = (\mathcal{S}, \mathcal{M}, A_1, B_6, \mathcal{T}, \mathcal{M}')$ of width w and expansion α, whose corresponding graph G_Π contains at most n vertices, where $n > c_0$ for some large enough constant c_0, and its maximum vertex degree is bounded by d. In order to simplify the notation, we denote G_Π by G. We also use the following parameter: $\rho = 2^{16} \left\lfloor \frac{d^3 n \log n}{\alpha^2 w} \right\rfloor$.

We are also given a graph $H = (U, F)$ of maximum degree 3, with $|U| \le \frac{w^2 \alpha^2}{2^{19} d^4 n \log n} \le \frac{w}{8d\rho}$.

Our goal is to find a model of H in G. Our algorithm consists of three steps. In the first step, we associate with each vertex $u \in U$, a subset X_u of vertices of W_2, such that $W_2[X_u]$ is a connected graph. This defines the embeddings of the vertices of H into G for the model of H that we are computing. In the second step, we embed all but a small fraction of the edges of H into G''_Π, and in the last step, we embed the remaining edges of H into G'_Π. We now describe each step in detail.

Step 1: Embedding the Vertices of H. In this step we compute an embedding of every vertex of H into a connected subgraph of W_2. Recall that graph W_2 is the union of the graphs S_2 and T_2, and the matching $\mathcal{M}'_2$, connecting the vertices of $B_2 \subseteq V(S_2)$ to the vertices of $C_2 \subseteq V(T_2)$, where $|B_2| = |C_2| = w$. We use the following simple observation, that was used extensively in the literature (often under the name of "grouping technique") (see e.g. [CKS05, RZ10, And10, Chu16b]). The proof is deferred to Section D of the Appendix.

Observation 3.2 *There is an efficient algorithm that, given a connected graph $\hat{G}$ with maximum vertex degree at most d, an integer $r \ge 1$, and a subset $R \subseteq V(\hat{G})$ of vertices of $\hat{G}$ with $|R| \ge r$, computes a collection $\{V_1, \ldots, V_r\}$ of r mutually disjoint subsets of $V(\hat{G})$, such that:*

- *For each $i \in [r]$, the induced graph $\hat{G}[V_i]$ is connected; and*
- *For each $i \in [r]$, $|V_i \cap R| \ge \lfloor |R|/(dr) \rfloor$.*

We apply the above observation to the graph T_2, together with vertex set $R = C_2$ and parameter $r = \left\lfloor \frac{w}{8d\rho} \right\rfloor$. Let $\mathcal{U}$ be the resulting collection of r subsets of vertices of T_2. Recall that for each set $V_i \in \mathcal{U}$, $|V_i \cap C_2| \ge \left\lfloor \frac{|C_2|}{dr} \right\rfloor \ge \left\lfloor \frac{w}{d\lfloor w/8d\rho \rfloor} \right\rfloor \ge 3\rho$. Since $|U| \le \frac{w}{8d\rho}$, we can choose $|U|$ distinct sets $V_1, \ldots, V_{|U|} \in \mathcal{U}$. We also denote $U = \left\{u_1, \ldots, u_{|U|}\right\}$. Finally, for each $1 \le i \le |U|$, we let $E^i \subseteq \mathcal{M}'_2$ be the subset of edges that have an endpoint in V_i, and we let B_2^i be the subset of vertices of B_2 that serve as endpoints of the edges in E^i. Since $|V_i \cap C_2| \ge 3\rho$, $|B_2^i| \ge 3\rho$ for all i. We are now ready to define the embeddings of the vertices of H into G. For each $1 \le i \le |U|$, we let $f(u_i) = G[B_2^i \cup V_i]$. Notice that for all $1 \le i \le |U|$, $f(u_i)$ is a connected graph, and for all $1 \le i < j \le |U|$, $f(u_i) \cap f(u_j) = \emptyset$. In the remaining steps, we focus on embedding the edges of H into G, such that the resulting paths are internally disjoint from $B_2 \cup T_2$.

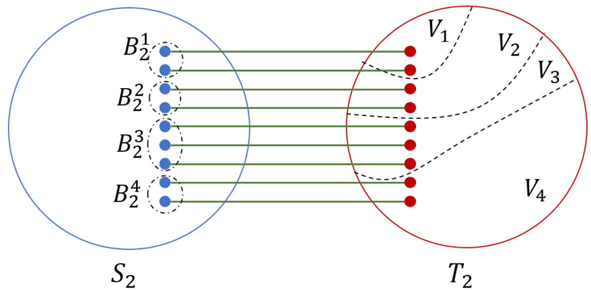


Figure 4: A sketch of the partition of T_2 and B_2. Vertices of B_2 and C_2 are shown in blue and red respectively.

Step 2: Routing in G''_Π. Consider some vertex $u_i \in U$, its corresponding graph $f(u_i)$, and the set $B_2^i \subseteq B_2$ of vertices that lie in $f(u_i)$; recall that $|B_2^i| \geq 3\rho$. Recall that the maximum vertex degree in H is at most 3. For every edge $e \in \delta_H(u_i)$, we select an arbitrary subset $B_2^i(e) \subseteq B_2^i$ of ρ vertices, so that all resulting sets $\left\{B_2^i(e)\right\}_{e\in\delta_H(u_i)}$ are mutually disjoint.

Recall that graph G_Π contains a perfect matching $\mathcal{M}_2$ between the vertices of B_2 and the vertices of A_3. We let $\hat{E}^i \subseteq \mathcal{M}_2$ be the subset of edges whose endpoints lie in B_2^i, and denote by $A_3^i \subseteq A_3$ the set of endpoints of the edges of E^i that lie in A_3. For every edge $e \in \delta(v_i)$, we let $A_3^i(e) \subseteq A_3^i$ be the set of ρ vertices that are connected to the vertices of $B_2^i(e)$ with an edge of $\mathcal{M}_2$. Clearly, all resulting vertex sets $\left\{A_3^i(e)\right\}_{e\in\delta_H(u_i)}$ are mutually disjoint. Let $A'_3 = \bigcup_{u_i\in U}\bigcup_{e\in\delta_H(u_i)} A_3^i(e)$, and notice that

$$|A'_3| \leq 3\rho \cdot |U| \leq 3\rho \cdot \frac{w}{8d\rho} = \frac{3w}{8d} \leq \frac{w}{2}.$$

The following lemma, whose proof is deferred to Section 4, allows us to embed a large number of edges of H in G''_Π.

Lemma 3.3 *There is an efficient algorithm, that, given a Path-of-Expanders System $\Pi = (\mathcal{S}, \mathcal{M}, A_1, B_6, \mathcal{T}, \mathcal{M}')$ of expansion α and width w, where $0 < \alpha < 1$ and w is an integral multiple of 4, whose corresponding graph G_Π contains at most n vertices and has maximum vertex degree at most d, together with a subset $A'_3 \subseteq A_3$ of at most $w/2$ vertices, and a collection $\left\{A_3^1, \ldots, A_3^{2r}\right\}$ of mutually disjoint subsets of A'_3 of cardinality $\rho = 2^{16}\left\lfloor\frac{d^3 n\log n}{\alpha^2 w}\right\rfloor$ each, where $r > \frac{w\alpha^2(\log\log n)^2}{d^3\log^3 n}$, returns a partition $\mathcal{I}', \mathcal{I}''$ of $\{1,\ldots,r\}$, and a set $\mathcal{P}^* = \left\{P_j^* \mid j \in \mathcal{I}'\right\}$ of disjoint paths in G''_Π, such that for each $j \in \mathcal{I}'$, path P_j^* connects A_3^j to A_3^{j+r}, and $|\mathcal{I}''| \leq \frac{w\alpha^2(\log\log n)^2}{d^3\log^3 n}$.*

We obtain the following immediate corollary of the lemma.

Corollary 3.4 *There is an efficient algorithm to compute a partition (F_1, F_2) of the set F of edges of H, and for each edge $e = (u_i, u_j) \in F_1$, a path P_e^* in graph G''_Π, connecting a vertex of $A_3^i(e)$ to a vertex of $A_3^j(e)$, such that all paths in set $\mathcal{P}_1^* = \{P^*(e) \mid e \in F_1\}$ are disjoint, and $|F_2| \leq \frac{w\alpha^2(\log\log n)^2}{d^3\log^3 n}$.*

Proof: By appropriately ordering the collection $\left\{A_3^i(e) \mid u_i \in U, e \in \delta_H(e)\right\}$ of vertex subsets, and applying Lemma 3.3 to the resulting sequence of subsets of A'_3, we obtain a set $F_1 \subseteq F$ of edges of

H, and for each edge $e=(u_i,u_j)\in F_1$, a path P^*_e, connecting a vertex of $A^i_3(e)$ to a vertex of $A^j_3(e)$ in graph G''_Π, such that all paths in set $\mathcal{P}^*_1=\{P^*(e)\mid e\in F_1\}$ are disjoint. Let $F_2=F\setminus F_1$. From Lemma 3.3, $|F_2|\leq \frac{w\alpha^2(\log\log n)^2}{d^3\log^3 n}$. □

For each edge $e=(u_i,u_j)\in F_1$, we extend the path P^*_e to include the two edges of $\mathcal{M}_2$ incident to its endpoints, so that P^*_e now connects a vertex of B^i_2 to a vertex of B^j_2. Path P^*_e becomes the embedding $f(e)$ of e in the model f of H that we are constructing. For convenience, the resulting set of paths $\{P^*_e\mid e\in F_1\}$ is still denoted by $\mathcal{P}^*_1$. The paths in $\mathcal{P}^*_1$ remain disjoint from each other; they are internally disjoint from W_2, and completely disjoint from W_1 (see Figure 5).

Step 3: Routing in G'_Π. In this step we complete the construction of a minor of H in G, by embedding the edges of F_2. The main tool that we use is the following lemma, whose proof is deferred to Section 5.

Lemma 3.5 *There is a universal constant c, and an efficient algorithm that, given a Path-of-Expanders System $\Pi=(\mathcal{S},\mathcal{M},A_1,B_6,\mathcal{T},\mathcal{M}')$ of expansion α and width w, such that the corresponding graph G_Π contains at most n vertices and has maximum vertex degree at most d, computes a subset $B'_1\subseteq B_1$ of at least $\frac{cw\alpha^2}{d^3\log^2 n}$ vertices, such that the following holds. There is an efficient randomized algorithm, that given any matching $\mathcal{M}^*$ over the vertices of B'_1, with high probability returns a set $\mathcal{P}$ of disjoint paths in W_1, routing $\mathcal{M}^*$.*

We now conclude the last step using the above lemma. Let $B'_1\subseteq B_1$ be the subset of at least $\frac{cw\alpha^2}{d^3\log^2 n}$ vertices, computed by algorithm from Lemma 3.5. Let $A'_2\subseteq A_2$ be the set of all the vertices connected to the vertices of B'_1 by the edges of the matching $\mathcal{M}_1$. Observe that $|A'_2|\geq 2|F_2|$, since:

$$2|F_2|\leq\frac{2w\alpha^2(\log\log n)^2}{d^3\log^3 n}\leq\frac{cw\alpha^2}{d^3\log^2 n}=|B'_1|=|A'_2|,$$

since we have assumed that n is sufficiently large. We let A''_2 be an arbitrary subset of $2|F_2|$ vertices of A'_2.

Recall that every vertex $u_i\in U$, and edge $e\in\delta_H(u_i)$, we have defined a subset $B^i_2(e)\subseteq B^i_2$ of vertices. We select an arbitrary representative vertex $b^i_2(e)\in B^i_2(e)$, and we let $B'_2=\left\{b^i_2(e)\mid u_i\in U, e\in\delta(u_i)\cap F_2\right\}$ be the resulting set of representative vertices, so that $|B'_2|=2|F|$.

Since $(A_2\cup B_2)$ are well-linked in S_2, there is a set $\mathcal{Q}_2$ of $2|F_2|$ disjoint paths in S_2, connecting every vertex of B'_2 to some vertex of A''_2, such that the paths in $\mathcal{Q}_2$ are internally disjoint from $A_2\cup B_2$. For each vertex $b^i_2(e)\in B'_2$, let $a^i_2(e)\in A''_2$ be the corresponding endpoint of the path of $\mathcal{Q}_2$ that originates at $b^i_2(e)$ (see Figure 6). Let $b^i_1(e)\in B'_1$ be the vertex of B_1 that is connected to $a^i_2(e)$ with an edge from $\mathcal{M}_1$. We can now naturally define a matching $\mathcal{M}^*$ over the vertices of B'_1, where for every edge $e=(u_i,u_j)\in F_2$, we add the pair $(b^i_1(e),b^j_1(e))$ of vertices to the matching. From Lemma 3.5, with high probability we obtain a collection $\mathcal{P}^*_2=\{P^*_e\mid e\in F_2\}$ of disjoint paths in W_1, such that, for every edge $e=(u_i,u_j)\in F_2$, the corresponding path P^*_e connects $b^i_1(e)$ to $b^j_1(e)$. We extend this path to connect the vertex $b^i_2(e)$ to the vertex $b^j_2(e)$, by using the edges of $\mathcal{M}_1$ that are incident to $b^i_1(e)$ to $b^j_1(e)$, and the paths of $\mathcal{Q}_2$ that are incident to $a^i_2(e)$ to $a^j_2(e)$. Notice that the resulting extended paths are internally disjoint from B_2, and are completely disjoint from $T_2\cup G''_\Pi$. We now embed each edge $e\in F_2$ into the path P^*_e, that is, we set $f(e)=P^*_e$. This completes the construction of the model of H in G_Π, except for the proofs of Lemmas 3.3 and 3.5, that are provided in Sections 4 and 5, respectively.

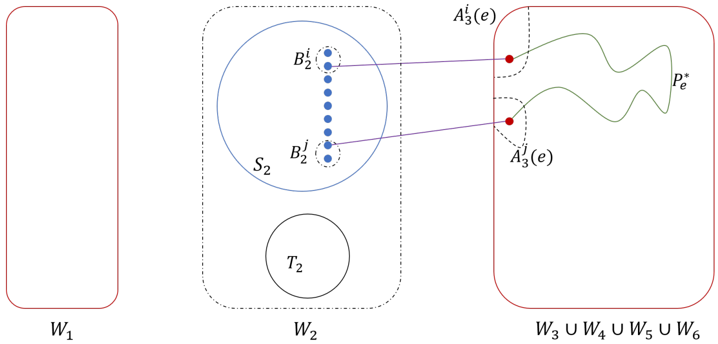


Figure 5: An illustration of a path $P_e^* \in \mathcal{P}_1^*$ routing an edge $e = (u_i, u_j) \in F_1$. Dashed boundaries represent the labeled subsets.

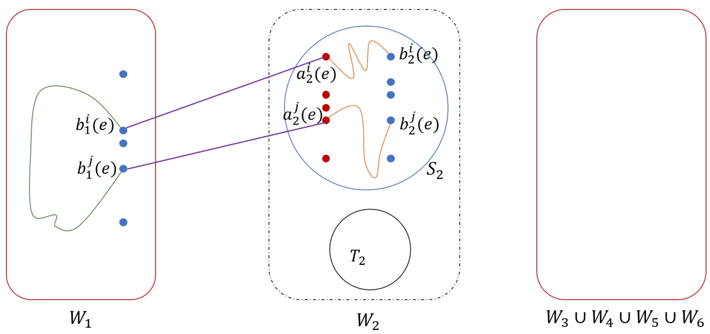


Figure 6: An illustration of the path $P_e^* \in \mathcal{P}_2^*$ connecting $e = (u_i, u_j) \in F_2$.

4 Routing in G''_{II}

This section is dedicated to the proof of Lemma 3.3. We define a new combinatorial object, called a Duo-of-Expanders System.

Definition 7 *A Duo-of-Expanders System of width w, expansion α (see Figure 7) consists of:*

- *two disjoint graphs T_1, T_2, each of which is an α-expander;*
- *a set X of w vertices that are disjoint from $T_1 \cup T_2$, and three subsets $D_0, D_1 \subseteq V(T_1)$ and $D_2 \subseteq V(T_2)$ of w vertices each, where all three subsets are disjoint; and*
- *a complete matching $\tilde{\mathcal{M}}$ between the vertices of X and the vertices of D_0, and a complete matching $\tilde{\mathcal{M}}'$ between the vertices of D_1 and the vertices of D_2, so $|\tilde{\mathcal{M}}| = |\tilde{\mathcal{M}}'| = w$.*

We denote the Duo-of-Expanders System by $\mathcal{D} = (T_1, T_2, X, \tilde{\mathcal{M}}, \tilde{\mathcal{M}}')$. The set X of vertices is called the backbone *of $\mathcal{D}$. Let $G_{\mathcal{D}}$ be the graph corresponding to the Duo-of-Expanders System $\mathcal{D}$, so $G_{\mathcal{D}}$ is the union of graphs T_1, T_2, the set X of vertices, and the set $\tilde{\mathcal{M}} \cup \tilde{\mathcal{M}}'$ of edges.*

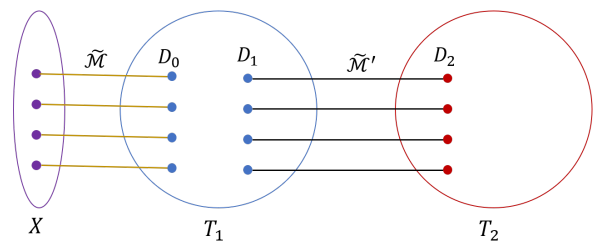


Figure 7: An illustration of the Duo-of-Expanders System.

Similarly to Path-of-Expanders System, given a graph G, we say that it contains a Duo-of-Expanders System $\mathcal{D}$ as a minor iff $G_{\mathcal{D}}$ is a minor of G.

The following lemma is central to the proof of Lemma 3.3.

Lemma 4.1 *There is an efficient algorithm that, given a Duo-of-Expanders System $\mathcal{D}$ of width $w/4$ and expansion α, for some $0 < \alpha < 1$, such that the corresponding graph $G_{\mathcal{D}}$ contains at most n vertices and has maximum vertex degree at most d, together with a collection $\{X_1, \dots, X_{2r}\}$ of mutually disjoint subsets of the backbone X of cardinality $\sigma = 2^{15}\left\lfloor \frac{d^3 n \log n}{\alpha^2 w}\right\rfloor$ each, where $r > \frac{w\alpha^2(\log\log n)^2}{d^3\log^3 n}$, returns a partition $\mathcal{I}', \mathcal{I}''$ of $\{1, \dots, r\}$, and for each $j \in \mathcal{I}'$, a path P_j connecting a vertex of X_j to a vertex of X_{j+r} in $G_{\mathcal{D}}$, such that the paths in set $\mathcal{P} = \{P_j \mid j \in \mathcal{I}'\}$ are disjoint, and $|\mathcal{I}''| \leq r \cdot \frac{\log\log n}{\log n}$.*

We defer the proof of Lemma 4.1 to Section 4.1, after we complete the proof of Lemma 3.3 using it. Recall that we are given a Path-of-Expanders System $\Pi = (\mathcal{S}, \mathcal{M}, A_1, B_6, \mathcal{T}, \mathcal{M}')$, together with its corresponding graph G_Π. Recall that we are also given a subset $A_3' \subseteq A_3$ of at most $w/2$ vertices, and a partition of A_3' into $2r$ disjoint subsets $A_3^1, \dots, A_3^{2r}$, of cardinality $\rho = 2^{16}\left\lfloor \frac{d^3 n \log n}{\alpha^2 w}\right\rfloor$ each, where $r \geq \frac{w\alpha^2(\log\log n)^2}{d^3\log^3 n}$. For each $1 \leq i \leq 2r$, we arbitrarily partition A_3^i into two subsets, W_1^i, W_2^i, of cardinality $\rho/2$ each (note that ρ is an even integer). Let $W_1 = \bigcup_{i=1}^{2r} W_1^i$ and let $W_2 = \bigcup_{i=1}^{2r} W_2^i$. Note that $|W_1|, |W_2| \leq |A_3'|/2 \leq w/4$. We add arbitrary vertices of $A_3 \setminus A_3'$ to W_1 and W_2, until each of them contains $w/4$ vertices (recall that $w/4$ is an integer), while keeping them disjoint. The vertices of $A_3 \setminus (W_1 \cup W_2)$ are then arbitrarily partitioned into two subsets, Y_1 and Y_2, of cardinality $w/4$ each.

Next, we show that graph G''_Π contains two disjoint Duo-of-Expanders Systems as minors. We will then use Lemma 4.1 in each of the two Duo-of-Expanders Systems in turn in order to obtain the desired routing.

Claim 4.2 *There is an efficient algorithm to compute two disjoint subgraphs, $G^{(1)}$ and $G^{(2)}$ of G''_Π, and for each $z \in \{1, 2\}$, to compute a model $f^{(z)}$ of a Duo-of-Expanders System $\mathcal{D}^{(z)} = (T_1^{(z)}, T_2^{(z)}, X^{(z)}, \tilde{\mathcal{M}}^{(z)}, (\tilde{\mathcal{M}}')^{(z)})$ of width $w/4$ and expansion α in $G^{(z)}$, such that the corresponding graph $G_{\mathcal{D}^{(z)}}$ has maximum vertex degree at most d, and for every vertex $w \in W_z$, there is a distinct vertex $v(w)$ in the backbone $X^{(z)}$, such that $w \in f^{(z)}(v(w))$.*

Proof of Claim 4.2. From the definition of the Path-of-Expanders System, for $3 \leq j \leq 6$, the set $A_j \cup B_j$ of vertices is well-linked in S_j. Therefore, there is a set $\mathcal{P}_j$ of w node-disjoint paths in S_j, connecting A_j to B_j. By concatenating the path sets $\mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5, \mathcal{P}_6$, and the edge sets $\mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_5$, we obtain a collection $\mathcal{P}$ of w node-disjoint paths in G''_Π, connecting A_3 to B_6. We partition $\mathcal{P}$ into two subsets: set $\mathcal{P}^{(1)}$ contains all paths originating at the vertices of $W_1 \cup Y_1$, and set $\mathcal{P}^{(2)}$ contains all paths originating at the vertices of $W_2 \cup Y_2$.

We are now ready to define the two graphs $G^{(1)}$ and $G^{(2)}$. Graph $G^{(1)}$ is obtained from the union of the expanders T_3 and T_4, the paths of $\mathcal{P}^{(1)}$, and the edges of $\mathcal{M}_3' \cup \mathcal{M}_4'$ that have an endpoint lying on the paths of $\mathcal{P}^{(1)}$. Graph $G^{(2)}$ is defined similarly by using T_5, T_6, the paths of $\mathcal{P}^{(2)}$, and the edges of $\mathcal{M}'_5 \cup \mathcal{M}'_6$ that have an endpoint lying on the paths of $\mathcal{P}^{(2)}$. It is immediate to verify that the graphs $G^{(1)}$ and $G^{(2)}$ are disjoint.

It now remains to show that each of the resulting graphs contains a Duo-of-Expanders System as a minor, with the required properties. We show this for $G^{(1)}$; the proof for $G^{(2)}$ is symmetric. Our first step is to contract every path of $\mathcal{P}^{(1)}$ into a single vertex. For each such path $P \in \mathcal{P}^{(1)}$, let $w \in W_1 \cup Y_1$ be the first vertex of P. We denote the new vertex obtained by contracting P by $v(w)$. We let the backbone $X^{(1)}$ of the new Duo-of-Expanders System $\mathcal{D}^{(1)}$ be $X^{(1)} = \{v(w) \mid w \in W_1\}$, so $|X^{(1)}| = w/4$. We map every vertex $w \in W_1$ to the corresponding vertex $v(w)$ in the model of $G_{\mathcal{D}^{(1)}}$ that we are constructing in $G^{(1)}$; that is, we set $f^{(1)}(w) = v(w)$. We also map the two expanders $T_1^{(1)}, T_2^{(1)}$ of $\mathcal{D}^{(1)}$ to T_3 and T_4, respectively, by setting $T_1^{(1)} = T_3$ and $T_2^{(1)} = T_4$.

Consider some vertex $w \in W_1 \cup Y_1$ and the path $P \in \mathcal{P}^{(1)}$ originating from w. Let w' be the unique vertex of P that belongs to B_3, and let w'' be the unique vertex of P that belongs to B_4, in the original Path-of-Expanders System Π. Recall that there is an edge of $\mathcal{M}'_3$, connecting w' to some vertex $u_w \in C_3$, and there is an edge of $\mathcal{M}'_4$, connecting w'' to some vertex $u'_w \in C_4$. Therefore, there are edges $(v(w), u_w)$ and $(v(w), u'(w))$ in the new contracted graph.

We set $D_0^{(1)} = \{u_w \mid w \in W_1\}$, and we let $\tilde{\mathcal{M}}^{(1)} = \{(v(w), u_w) \mid w \in W_1\}$. We also set $D_1^{(1)} = \left\{u_y \mid y \in \hat{Y}_1\right\}$, and $D_1^{(2)} = \left\{u'_y \mid y \in \hat{Y}_1\right\}$. Observe that all three sets $D_0^{(1)}, D_1^{(1)}, D_2^{(1)}$ of vertices are disjoint, and they contain $w/4$ vertices each. It now remains to define the set $(\tilde{\mathcal{M}}')^{(1)}$ of edges, that connect vertices of $D_1^{(1)}$ and $D_2^{(1)}$. In order to do so, for every vertex $y \in Y_1$, we merge the two edges $(v(y), u_y)$ and $(v(y), u'_y)$ into a single edge, by contracting one of these two edges. The resulting edge is added to $(\tilde{\mathcal{M}}')^{(1)}$. It is easy to see that we have obtained a Duo-of-Expanders System $\mathcal{D}^{(1)}$, whose width is $w/4$ and expansion α. It is easy to verify that the maximum vertex degree in the corresponding graph $G_{\mathcal{D}^{(1)}}$ is bounded by d. Notice that for every vertex $w \in W_1$, there is a distinct vertex $v(w) \in X^{(1)}$, such that $w \in f^{(1)}(v(w))$. $\square$

We apply Lemma 4.1 to $\mathcal{D}^{(1)}$, together with vertex sets $W_1^1, \ldots, W_1^{2r}$, each of which now contains $\rho/2 = 2^{15}\left\lfloor \frac{d^3 n \log n}{\alpha^2 w} \right\rfloor$ vertices obtaining a partition $(\mathcal{I}', \mathcal{I}'')$ of $\{1, \ldots, r\}$, together with a set $\mathcal{P}_1 = \{P_j \mid j \in \mathcal{I}'\}$ of disjoint paths in $G_{\mathcal{D}^{(1)}}$, such that for all $j \in \mathcal{I}'$ path P_j connects a vertex of W_1^j to a vertex of W_1^{r+j}, and $|\mathcal{I}''| \leq r \cdot \frac{\log\log n}{\log n}$. Since $G_{\mathcal{D}^{(1)}}$ is a minor of $G^{(1)}$, it is immediate to obtain a collection $\mathcal{P}'_1 = \left\{P'_j \mid j \in \mathcal{I}'\right\}$ of disjoint paths in $G^{(1)}$, such that for all $j \in \mathcal{I}'$ path P'_j connects a vertex of A_3^j to a vertex of A_3^{j+r}.

If $|\mathcal{I}''| \leq \frac{w\alpha^2 (\log\log n)^2}{d^3 \log^3 n}$, then we terminate the algorithm, and return the set $\mathcal{P}'$ of paths, together with the partition $(\mathcal{I}', \mathcal{I}'')$ of $\mathcal{I}$. Next, we denote $|\mathcal{I}''| = r'$, and we assume that $r' > \frac{w\alpha^2 (\log\log n)^2}{d^3 \log^3 n}$.

We apply Lemma 4.1 to $\mathcal{D}^{(2)}$, together with vertex sets $\left\{W_2^j, W_2^{j+r} \mid j \in \mathcal{I}''\right\}$, that are appropriately ordered. We then obtain a partition $\mathcal{I}_1, \mathcal{I}_2$ of $\mathcal{I}''$, and a set $\mathcal{P}_2 = \{P_j \mid j \in \mathcal{I}_1\}$ of disjoint paths in $G_{\mathcal{D}^{(2)}}$, such that for each $j \in \mathcal{I}_2$ path P_j connects a vertex of W_2^j to a vertex of W_2^{j+r}, and $|\mathcal{I}_2| \leq r' \cdot \frac{\log\log n}{\log n} \leq r \cdot \frac{(\log\log n)^2}{\log^2 n}$. As before, since $G_{\mathcal{D}^{(2)}}$ is a minor of $G^{(2)}$, it is immediate to obtain a collection $\mathcal{P}'_2 = \left\{P'_j \mid j \in \mathcal{I}_1\right\}$ of disjoint paths in $G^{(2)}$, such that for all $j \in \mathcal{I}_1$ path P'_j connects a vertex of A_3^j to a vertex of A_3^{j+r}.

We return the partition $(\mathcal{I}' \cup \mathcal{I}_1, \mathcal{I}_2)$ of $\{1, \ldots, r\}$, together with the set $\mathcal{P}'_1 \cup \mathcal{P}'_2$ of paths. Since the graphs $G^{(1)}$ and $G^{(2)}$ are disjoint, all paths in $\mathcal{P}'_1 \cup \mathcal{P}'_2$ are disjoint. It now only remains to show that $|\mathcal{I}_2| \leq \frac{w\alpha^2(\log\log n)^2}{d^3 \log^3 n}$.

Recall that the set A'_3 of at most $w/2$ vertices is partitioned into $2r$ subsets of cardinality $\rho = 2^{16}\left\lfloor \frac{d^3 n \log n}{\alpha^2 w} \right\rfloor$ each. Therefore:

$$
\begin{aligned}
r &\leq \frac{w}{4\rho} \\
&= \frac{w}{2^{18}\left\lfloor d^3 n \log n/(\alpha^2 w)\right\rfloor} \\
&\leq \frac{w^2\alpha^2}{2^{17} d^3 n \log n} \\
&\leq \frac{w\alpha^2}{2^{17} d^3 \log n}.
\end{aligned}
$$

Therefore, $|\mathcal{I}_2| \leq r \cdot \frac{(\log\log n)^2}{\log^2 n} \leq \frac{w\alpha^2(\log\log n)^2}{d^3 \log^3 n}$, as required.

4.1 Routing in Duo-of-Expanders — Proof of Lemma 4.1

The goal of this section is to prove Lemma 4.1. The proof is inspired by the algorithm of Frieze [Fri01] for routing a large set of demand pairs in an expander graph via edge-disjoint paths. Recall that we are given a Duo-of-Expanders System $\mathcal{D}$ of width $w/4$ and expansion α, for some $0 < \alpha < 1$, such that the maximum vertex degree in the corresponding graph $G_{\mathcal{D}} = (V, E)$ is at most d, and $|V| \leq n$. We are also given mutually disjoint subsets $\{X_1, \ldots, X_{2r}\}$ of the backbone X, of cardinality $\sigma = 2^{15}\left\lfloor \frac{d^3 n \log n}{w\alpha^2} \right\rfloor$ each, where $r > \frac{w\alpha^2(\log\log n)^2}{d^3 \log^3 n}$. In particular, since $|X| = w/4$, we get that $2r\sigma \leq w/4$, and so $r \leq \frac{w}{8\sigma} \leq \frac{w}{8\cdot 2^{15}\lfloor d^3 n \log n/(w\alpha^2)\rfloor} \leq \frac{w^2\alpha^2}{2^{17} d^3 n \log n}$. Therefore, we obtain the following bounds on r that we will use throughout the proof:

$$
\frac{w\alpha^2}{d^3} \cdot \frac{(\log\log n)^2}{\log^3 n} < r \leq \frac{w^2\alpha^2}{2^{17} d^3 n \log n}. \tag{1}
$$

For convenience, we will denote $G_{\mathcal{D}}$ by G for the rest of this subsection.

We will iteratively construct the set $\mathcal{P}$ of disjoint paths in G, where for each path $P \in \mathcal{P}$, there is some index $j \in \{1, \ldots, r\}$, such that P connects X_j to X_{j+r}. Whenever a path P is added to $\mathcal{P}$, we delete all vertices of P from G. Throughout the algorithm, we say that an index $j \in [r]$ is *settled* iff there is a path $P_j \in \mathcal{P}$ connecting X_j to X_{j+r}, and otherwise we say that it is *not settled.* We use a parameter $\gamma = 512nd^2/w\alpha$. We say that a path P in G is *permissible* iff P contains at most $\gamma \log\log n$ nodes of T_1 and at most $\gamma \log n$ nodes of T_2.

The Algorithm. Start with $\mathcal{P} = \emptyset$. While there is an index $j \in [r]$ and a permissible path P^*_j in the current graph G such that:

- j is not settled;
- P^*_j connects X_j to X_{j+r}; and

- P^*_j is internally disjoint from X:

add P^*_j to $\mathcal{P}$ and delete all vertices of P^*_j from G.

In order to complete the proof of Lemma 4.1, it is enough to show that, when the algorithm terminates, at most $\frac{r\log\log n}{\log n}$ indices $j \in [r]$ are not settled. Assume for contradiction that this is not true. Let $\mathcal{P}$ be the path set obtained at the end of the algorithm, and let $\tilde{V} = V(\mathcal{P})$ be the set of vertices participating in the paths of $\mathcal{P}$. We further partition $\tilde{V}$ into three subsets: $\tilde{V}_1 = \tilde{V} \cap V(T_1)$; $\tilde{V}_2 = \tilde{V} \cap V(T_2)$; and $\tilde{X} = \tilde{V} \cap X$. Note that, since $|\mathcal{P}| \le r$, we are guaranteed that $|\tilde{V}_1| \le \gamma r \log\log n$; $|\tilde{V}_2| \le \gamma r \log n$, and, since we have assumed that $|\mathcal{P}| \le r(1-\log\log n/\log n)$, and all paths in $\mathcal{P}$ are internally disjoint from X, we get that $|\tilde{X}| \le 2r - \frac{2r\log\log n}{\log n}$.

We now proceed as follows. First, we show that $T_1 \setminus \tilde{V}_1$ and $T_2 \setminus \tilde{V}_2$ both contain very large $\alpha/4$-expanders. We also show that there is a large number of edges in $\tilde{\mathcal{M}}'$ that connect these two expanders. This will be used to show that there must still be a permissible path P^*_j, connecting two sets X_j and X_{j+r} for some index j that is not settled yet, leading to a contradiction. We start with the following claim that allows us to find large expanders in $T_1 \setminus \tilde{V}_1$ and $T_2 \setminus \tilde{V}_2$.

Claim 4.3 *Let T be an α-expander with maximum vertex degree at most d, and let Z be any subset of vertices of T. Then there is an $\alpha/4$-expander $T' \subseteq T \setminus Z$, with $|V(T')| \ge |V(T)| - \frac{4d|Z|}{\alpha}$.*

The proof of Claim 4.3 follows immediately from Claim 2.3, by letting E' be the set of all edges incident to the vertices of Z. The following corollary follows immediately from Claim 4.3

Corollary 4.4 *There is a subgraph $T'_1 \subseteq T_1 \setminus \tilde{V}_1$ that is an $\alpha/4$-expander, and $|V(T_1) \setminus V(T'_1)| \le 4dr\gamma \log\log n/\alpha$. Similarly, there is a subgraph $T'_2 \subseteq T_2 \setminus \tilde{V}_2$ that is an $\alpha/4$-expander, and $|V(T_2) \setminus V(T'_2)| \le 4dr\gamma \log n/\alpha$.*

Let $R_1 = V(T_1) \setminus V(T'_1)$ and let $R_2 = V(T_2) \setminus V(T'_2)$. We refer to the vertices of R_1 and R_2 as the vertices that were *discarded* from T_1 and T_2, respectively. The vertices that belong to T'_1 and T'_2 are called *surviving* vertices. It is easy to verify that $|R_1|, |R_2| \le w/64$. Indeed, observe that $|R_1|, |R_2| \le 4dr\gamma \log n/\alpha$. Since, from Equation (1), $r \le \frac{w^2\alpha^2}{2^{17}d^3 n \log n}$, we get that altogether:

$$|R_1|, |R_2| \le \frac{4dr\gamma \log n}{\alpha} \le \frac{\gamma w^2 \alpha}{2^{15} d^2 n} \le \frac{w}{64},$$

since $\gamma = 512nd^2/w\alpha$.

Recall that the Duo-of-Expanders $\mathcal{D}$ contains a matching $\tilde{\mathcal{M}}'$ between the set $D_1 \subseteq V(T_1)$ of $w/4$ vertices and the set $D_2 \subseteq V(T_2)$ of $w/4$ vertices. Next, we show that there are large subsets $D'_1 \subseteq D_1$ and $D'_2 \subseteq D_2$ of surviving vertices, such that a subset of $\tilde{\mathcal{M}}'$ defines a complete matching between them.

Observation 4.5 *There are two sets $D'_1 \subseteq D_1$ and $D'_2 \subseteq D_2$ containing at least $w/16$ vertices each, and a subset $\hat{\mathcal{M}} \subseteq \tilde{\mathcal{M}}'$ of edges, such that $\hat{\mathcal{M}}$ is a complete matching between D'_1 and D'_2.*

Proof: Let $\hat{D}_1 = D_1 \setminus R_1$. Since $|R_1| \le w/64$, $|\hat{D}_1| \ge w/8$. Let $\hat{\mathcal{M}}' \subseteq \mathcal{M}'$ be the set of edges whose endpoints lie in $\hat{D}_1$, and let $\hat{D}_2 \subseteq D_2$ be the set of vertices that serve as endpoints for the edges in

$\hat{\mathcal{M}}'$, so $|\hat{D}_2| \geq w/8$. Finally, let $D'_2 = D_2 \setminus R_2$, so $|D'_2| \geq w/8 - |R_2| \geq w/16$. We let $\hat{\mathcal{M}} \subseteq \hat{\mathcal{M}}'$ be the set of all edges incident to the vertices of D'_2, and we let D'_1 be the set of endpoints of these edges. □

Our second main tool is the following claim, that shows that for any pair of large enough sets of vertices in an expander, there is a short path connecting them. The proof uses standard methods and is deferred to Appendix.

Claim 4.6 *Let T be an α'-expander for some $0 < \alpha' < 1$, such that $|V(T)| \leq n$, and the maximum vertex degree in T is at most d. Let $Z, Z' \subseteq V(T)$ be two vertex subsets, with $|Z| = z$ and $|Z'| = z'$. Then there is a path in T, connecting a vertex of Z to a vertex of Z', whose length is at most $\frac{8d}{\alpha'}(\log(n/z) + \log(n/z'))$. In particular, for every pair v, v' of vertices in T, there is a path of length at most $16d \log n/\alpha'$ connecting v to v' in T.*

Let $J \subseteq \{1, \ldots, r\}$ be the set of indices that are not settled yet. From our assumption, $|J| \geq \frac{r \log\log n}{\log n}$. For every index $j \in J$, consider the corresponding sets X_j, X_{j+r} of vertices of X, and let Y_j, Y_{j+r} be the sets of vertices of D_0, that are connected to X_j and X_{j+r} via the matching $\tilde{\mathcal{M}}$. Let $Y'_j = Y_j \setminus R_1$ and let $Y'_{j+r} = Y_{j+r} \setminus R_1$ be the subsets of surviving vertices in Y_j and Y_{j+r} respectively. We say that index j is *bad* iff $|Y'_j| < \sigma/2$ or $|Y'_{j+r}| < \sigma/2$; otherwise we say that it is a *good index*. Recall that $|R_1| \leq 4dr\gamma \log\log n/\alpha$. Therefore, the total number of bad indices is at most:

$$\begin{aligned}\frac{2|R_1|}{\sigma} &\leq \frac{8dr\gamma \log\log n}{\alpha \cdot 2^{15} \lfloor d^3 n \log n/(w\alpha^2) \rfloor} \\ &\leq \frac{w\alpha r\gamma \log\log n}{2^{11} d^2 n \log n} \\ &\leq \frac{r \log\log n}{4 \log n} \cdot \frac{w\alpha\gamma}{512 d^2 n} \\ &\leq \frac{r \log\log n}{4 \log n},\end{aligned}$$

since $\gamma = 512nd^2/w\alpha$.

Let $J' \subseteq J$ be the set of all good indices, so $|J'| \geq \frac{r \log\log n}{2 \log n}$. We say that an index $j \in J'$ is *happy* iff there is a path $P_1(j)$ in T'_1, of length at most $(\gamma \log\log n)/4$, connecting a vertex of Y'_j to a vertex of D'_1, and there is a path $P_2(j)$ in T'_1, of length at most $(\gamma \log\log n)/4$, connecting a vertex of Y'_{j+r} to a vertex of D'_1. The following claim will finish the proof of Lemma 3.3.

Claim 4.7 *At least one index of J' is happy.*

Assume first that the claim is correct. Consider the paths $P_1(j)$ and $P_2(j)$ in T', given by Claim 4.7, and assume that path $P_1(j)$ connects a vertex $v \in Y'_j$ to a vertex $v' \in D'_1$. Let $v'' \in D'_2$ be the vertex connected to v' by an edge of $\hat{\mathcal{M}}$, that we denote by e_v. Similarly, assume that path $P_2(j)$ connects a vertex $u \in Y'_j$ to a vertex $u' \in D'_1$. Let $u'' \in D'_2$ be the vertex connected to u' by an edge of $\hat{\mathcal{M}}$, that we denote by e_u. From Claim 4.6, there is a path P in T'_2, of length at most $64d \log n/\alpha < \gamma \log n$, connecting v'' to u''. By combining $P_1(j), e_v, P', e_u, P_2(j)$, together with the edges of $\tilde{\mathcal{M}}$ incident to u and v, we obtain an admissible path, connecting a vertex of X_j to a vertex of X_{j+r}, a contradiction. It now remains to prove Claim 4.7.

Proof of 4.7. We say that a vertex v of $D_0 \cap V(T'_1)$ is *happy* iff there is a path in T'_1, of length at most $(\gamma \log\log n)/4$, connecting v to a vertex of D'_1. Assume for contradiction that the claim is false. Then for each good index j, either all vertices of Y'_j are unhappy, or all vertices of Y'_{j+r} are unhappy.

Let $Z \subseteq D_0 \cap V(T_1')$ be the set of all unhappy vertices. Since $|Y_j'|, |Y_{j+1}'| \geq \sigma/2$, and $|J'| \geq \frac{r \log\log n}{2 \log n}$, we get that:

$$\begin{aligned}
|Z| &\geq \frac{r \log\log n}{2 \log n} \cdot \frac{\sigma}{2} \\
&\geq \frac{w\alpha^2 (\log\log n)^3}{2d^3 \log^4 n} \cdot 2^{14} \cdot \left\lfloor \frac{d^3 n \log n}{w\alpha^2} \right\rfloor \\
&\geq \frac{2^{12} n (\log\log n)^3}{\log^3 n}.
\end{aligned}$$

Let $Z' = D_1'$, so $|Z'| \geq w/16$. From Claim 4.6, there is a path in T_1', connecting a vertex of Z to a vertex of Z', of length at most: $\frac{32d}{\alpha}(\log(n/|Z|) + \log(n/|Z'|)) \leq \frac{32d}{\alpha}\left(\log\left(\frac{\log^3 n}{2^{13}(\log\log n)^3}\right) + \log(\frac{16n}{w})\right) \leq \frac{32d}{\alpha}(3 \log\log n + \log(\frac{16n}{w})) \leq (\gamma \log\log n)/4$, since $\gamma = 512nd^2/(w\alpha)$. □

5 Routing in G'_Π

The goal of this section is to prove Lemma 3.5. We use the following lemma, whose proof uses standard techniques and is deferred to Section F of Appendix.

Lemma 5.1 *There is a universal constant c, and an efficient randomized algorithm, that, given graph $G = (V, E)$ with $|V| \leq n$, such that the maximum vertex degree in G is at most d and a parameter $0 < \alpha < 1$, together with a collection $\{C_1, \ldots, C_{2r}\}$ of mutually disjoint subsets of V of cardinality $q = \lceil cd^2 \log^2 n/\alpha^2 \rceil$ each, computes one of the following:*

- *either a collection $\mathcal{Q} = \{Q_1, \ldots, Q_r\}$ of paths in G, where for each $1 \leq j \leq r$, path Q_j connects a vertex of C_j to a vertex of C_{r+j}, and with high probability the paths in $\mathcal{Q}$ are disjoint; or*
- *a cut (S, S') in G of sparsity less than α.*

Consider the subgraph W' of G_Π; recall that it consists of two graphs, S_1 and T_1, where S_1 is a connected graph and T_1 is an α-expander. Recall that S_1 contains a set B_1 of w vertices; T_1 contains a set C_1 of w vertices, and $\mathcal{M}_1'$ is a perfect matching between these two sets.

We let $q = \lceil cd \log^2 n/\alpha^2 \rceil$, where c is the constant from Lemma 5.1, and we let $r = \lfloor w/dq \rfloor = \Omega(w\alpha^2/d^3 \log^2 n)$. Observe that $q \leq \lfloor w/dr \rfloor$. We use Observation 3.2 to compute r connected subgraphs $S^1, \ldots, S^r$ of S_1, each of which contains at least $\lfloor w/dr \rfloor \geq q$ vertices of B_1. For $1 \leq i \leq r$, we denote $B^i = B_1 \cap V(S^i)$. We also let $\mathcal{M}^i \subseteq \mathcal{M}_1'$ be the set of edges incident to the vertices of B^i in $\mathcal{M}_1'$, and we let $C^i \subseteq C_1$ be the set of the endpoints of the edges of $\mathcal{M}^i$ that lie in C_1. Observe that for all $1 \leq i \leq 2r$, $|C^i| \geq q$. For each $1 \leq i \leq 2r$, we select an arbitrary vertex $b_i \in B^i$, and we let $B' = \{b_i \mid 1 \leq i \leq 2r\}$, so that $|B'| = 2r = \Omega(w\alpha^2/d^3 \log^2 n)$, as required.

Assume now that we are given an arbitrary matching $\mathcal{M}^*$ over the vertices of B'. By appropriately re-indexing the sets B^i, we can assume w.l.o.g. that $\mathcal{M}^* = \{(b_i, b_{r+i})\}_{i=1}^r$. Since T_1 is an α-expander, the algorithm of Lemma 5.1 computes a collection $\mathcal{Q} = \{Q_1, \ldots, Q_r\}$ of paths in T_1, where for each $1 \leq j \leq r$, path Q_j connects some vertex $c_j^* \in C^j$ to some vertex $c_{j+r}^* \in C^{j+r}$, and with high probability the paths in $\mathcal{Q}$ are disjoint.

Consider now some index $1 \leq j \leq 2r$. We let e_j be the unique edge of the matching $\mathcal{M}'_1$ incident to c^*_j, and we let $b^*_j \in B^j$ be the other endpoint of this edge. Since graph S^j is connected, and it contains both b_j and b^*_j, we can find a path P_j in S^j, connecting b_j to b^*_j. For each $1 \leq j \leq r$, let P^*_j be the path obtained by concatenating $P_j, e_j, Q_j, e_{j+r}, P_{j+r}$, and let $\mathcal{P}^* = \left\{P^*_j \mid 1 \leq j \leq r\right\}$. It is immediate to verify that, if the paths in $\mathcal{Q}$ are disjoint from each other, then so are the paths in $\mathcal{P}^*$, since all graphs in $\left\{S^j \mid 1 \leq j \leq 2r\right\}$ are disjoint from each other and from T_1. Moreover, for each $1 \leq j \leq r$, path P^*_j connects b_j to b_{j+r}, as required.

6 Constructing a Path-of-Expanders System

The goal of this section is to prove Theorem 2.4. The proof consists of three parts. In the first part, we construct an α'-expanding Path-of-Sets System of length 24 in G, for some α'. In the second part, we transform it into a Strong Path-of-Sets System of the same length. In the third and the final part, we turn the Strong Path-of-Sets System into a Path-of-Expanders System.

6.1 Part 1: Constructing an Expanding Path-of-Sets System

The main technical result of this section is the following theorem.

Theorem 6.1 *There is a constant $c_x > 3$, and a deterministic algorithm, that, given an n-vertex α-expander G with maximum vertex degree at most d, where $0 < \alpha < 1$, computes, in time $\text{poly}(n) \cdot \left(\frac{d}{\alpha}\right)^{O(\log(d/\alpha))}$ a partition (V', V'') of $V(G)$, such that $|V'|, |V''| \geq \frac{\alpha|V(G)|}{256d}$, and each graph $G[V'], G[V'']$ is an α^*-expander, for $\alpha^* \geq \left(\frac{\alpha}{d}\right)^{c_x}$.*

The main tool that we use in the proof of the theorem is the following lemma.

Lemma 6.2 *There is a constant c'_x, and deterministic algorithm, that, given an n-vertex α-expander G with maximum vertex degree at most d, where $0 < \alpha < 1$, computes, in time $\text{poly}(n) \cdot \left(\frac{d}{\alpha}\right)^{O(\log(d/\alpha))}$, a subset $V' \subseteq V(G)$ of vertices, such that $\frac{\alpha|V(G)|}{256d} \leq |V'| \leq \frac{\alpha|V(G)|}{8d}$, and $G[V']$ is an $\hat{\alpha}^*$-expander, for $\hat{\alpha}^* \geq \left(\frac{\alpha}{d}\right)^{c'_x}$.*

Proof: Given a graph G, we say that a partition (U', U'') of $V(G)$ is a *balanced cut* iff $|U'|, |U''| \geq |V(G)|/4$.

Our starting point is the following claim.

Claim 6.3 *There is an efficient algorithm that, given an n-vertex graph $G = (V, E)$, and a parameter β, returns one of the following:*

- *either a subset $V' \subseteq V$ of vertices, such that $n/2 \leq |V'| \leq 3n/4$ and $G[V']$ is an $\Omega(\frac{\beta^2}{d})$-expander;*
- *or a partition (S, T) of V with $|E_G(S, T)| < \beta \cdot \min\{|S|, |T|\}$.*

Proof: We start with an arbitrary balanced cut (U', U'') in G with $|U'| \geq |U''|$, and perform a number of iterations. In every iteration, we will either establish that $G[U']$ is an $\Omega(\frac{\beta^2}{d})$-expander, or compute the desired partition (S, T) of V, or find a new balanced cut (J', J'') in G with $|E(J', J'')| < |E(U', U'')|$. In the first two cases, we terminate the algorithm and return either $V' = U'$ (in the first case), or the cut (S,T) (in the second case). In the last case, we replace (U',U'') with (J',J''), and continue to the next iteration.

We now describe the execution of an iteration. Recall that we are given a balanced cut (U',U'') of G with $|U'| \geq |U''|$. If $|E(U',U'')| < \beta \cdot \min\{|U'|,|U''|\}$, then we return the cut $(S,T) = (U',U'')$ and terminate the algorithm. Therefore, we assume that $|E(U',U'')| \geq \beta \cdot \min\{|U'|,|U''|\}$. We apply the algorithm from Theorem 2.2 to graph $G[U']$, and consider the cut (S,T) of $G[U']$ computed by the algorithm. We then consider two cases. First, if $|E(S,T)| \geq \frac{\beta}{4}\min\{|S|,|T|\}$, then from Theorem 2.2, we are guaranteed that $G[U']$ is an $\Omega(\frac{\beta^2}{d})$-expander. We terminate the algorithm and return $V' = U'$.

We assume that $|E(S,T)| < \frac{\beta}{4}\min\{|S|,|T|\}$ from now on, and we assume w.l.o.g. that $|T| \leq |S|$. We consider again two cases. First, if $|E(T,U'')| \leq \frac{\beta}{2}|T|$, we define a new cut (S',T) in G, where $S' = S \cup U''$. We then get that $|T| \leq |S'|$, and moreover, $|E_G(S',T)| = |E_G(S,T)| + |E_G(U'',T)| < \beta|T|$. We return the cut (S',T) and terminate the algorithm.

The final case is when $|E(T,U'')| > \frac{\beta}{2}|T|$. In this case, we are guaranteed that $|E(T,U'')| > |E(S,T)|$. Therefore, if we consider the cut (J',J''), where $J' = S$ and $J'' = T \cup U''$, then (J',J'') is a balanced cut in G, and moreover:

$$|E(J',J'')| = |E(S,U'')| + |E(S,T)| < |E(S,U'')| + |E(T,U'')| = |E(U',U'')|.$$

We then replace (U',U'') with the new cut (J',J''), and continue to the next iteration. It is easy to verify that every iteration can be executed in time $\text{poly}(n)$. Since the number of the edges in the set $E(U',U'')$ decreases in every iteration, the number of iterations is also bounded by $\text{poly}(n)$. □

By combining Claim 6.3 with Observation 2.1, we obtain the following simple corollary.

Corollary 6.4 *There is an efficient algorithm that, given an n-vertex graph $G = (V,E)$ with maximum vertex degree at most d, and a parameter β, returns one of the following:*

- *either a subset $V' \subseteq V$ of vertices, such that $n/4 \leq |V'| \leq 3n/4$ and $G[V']$ is an $\Omega(\frac{\beta^2}{d})$-expander;*
- *or a* **balanced** *partition (S,T) of V with $|E_G(S,T)| < \beta \cdot \min\{|S|,|T|\}$.*

Proof: Throughout the algorithm, we maintain a set E' of edges of G that we remove from the graph, starting with $E' = \emptyset$, and a collection $\mathcal{G}$ of disjoint induced subgraphs of $G \setminus E'$, starting with $\mathcal{G} = \{G\}$. The algorithm continues as long as there is some graph $H \in \mathcal{G}$, with $|V(H)| > 3|V(G)|/4$. In every iteration, we select the unique graph $H \in \mathcal{G}$ with $|V(H)| > 3|V(G)|/4$, and apply Claim 6.3 to it, with the parameter $\beta/4$. If the outcome is a subset $V' \subseteq V(H)$ of vertices, such that $|V(H)|/2 \leq |V'| \leq 3|V(H)|/4$, and $H[V']$ is an $\Omega(\frac{\beta^2}{d})$-expander, then we return V': it is easy to verify that $n/4 \leq |V'| \leq 3n/4$, so V' is a valid output. Otherwise, we obtain a partition (S',T') of $V(H)$ with $|E(S',T')| < \frac{\beta}{4} \cdot \min\{|S'|,|T'|\}$. We add the edges of $E(S',T')$ to E', remove H from $\mathcal{G}$, and add $H[S']$ and $H[T']$ to $\mathcal{G}$ instead. If $|S'| < |T'|$, then our algorithm will never attempt to process the graph $H[S']$ again, so we *charge* the edges of $E(S',T')$ to the vertices of S', where every vertex of S' is charged fewer than $\beta/4$ units. The algorithm terminates when every graph $H \in \mathcal{G}$ has $|V(H)| \leq 3n/4$ (unless it terminates earlier with an expander). Notice that from our charging scheme, at the end of the algorithm, $|E'| < n\beta/4$. Moreover, using Observation 2.1, we can partition the final collection $\mathcal{H}$ of graphs into two subsets, $\mathcal{H}', \mathcal{H}''$, such that $\sum_{H \in \mathcal{H}'} |V(H)|, \sum_{H \in \mathcal{H}''} |V(H)| \geq n/4$. Letting $S = \bigcup_{H \in \mathcal{H}'} V(H)$ and $T = \bigcup_{H \in \mathcal{H}''} V(H)$, we obtain a balanced partition (S,T) of $V(G)$. Since $E(S,T) \subseteq E'$, we get that $|E(S,T)| < \frac{\beta n}{4} \leq \beta \cdot \min\{|S|,|T|\}$. □

We now turn to complete the proof of Lemma 6.2. We denote $|V(G)| = n$, and we let $n^* = \alpha|V(G)|/(8d)$. Our goal now is to compute a subset $V' \subseteq V(G)$ of vertices, with $n^*/32 \le |V'| \le n^*$, such that $G[V']$ is an $\hat{\alpha}^*$-expander, where $\hat{\alpha}^* \ge \left(\frac{\alpha}{d}\right)^{c'_x}$ for some constant c'_x. Our algorithm is recursive. Over the course of the algorithm, we will consider smaller and smaller sub-graphs of G, containing at least $n^*/4$ vertices each. For each such subgraph $G' \subseteq G$, we define its *level* $L(G')$ as follows. Let $n' = |V(G')|$. If $n' \le 4n^*/3$, then $L(G') = 0$; otherwise, $L(G') = \left\lceil \log_{4/3}(n'/n^*) \right\rceil$. Intuitively, $L(G')$ is the number of recursive levels that we will use for processing G'. Notice that, from the definition of n^*, $L(G) \le O(\log(d/\alpha))$. We use the following claim.

Claim 6.5 *There is a deterministic algorithm, that, given a subgraph $G' \subseteq G$, such that $|V(G')| \ge n^*/4$, and a parameter $0 < \beta < 1$, returns one of the following:*

- *Either a balanced cut (S,T) in G' with $|E_{G'}(S,T)| < \beta \cdot \min\{|S|,|T|\}$; or*
- *A subset $V' \subseteq V(G')$ of vertices of G', such that $n^*/32 \le |V'| \le n^*$, and $G'[V']$ is an $\hat{\beta}$-expander, for $\hat{\beta} \ge \Omega\left(\frac{\beta^2}{d \cdot 2^{10L(G')}}\right)$.*

The running time of the algorithm is $\mathrm{poly}(n) \cdot \left(\frac{256d}{\hat{\beta}}\right)^{L(G')}$.

We prove the claim below, after we complete the proof of Lemma 6.2 using it. We apply Claim 6.5 to the input graph G and the parameter α. Since G is an α-expander, we cannot obtain a cut (S,T) in G with $|E(S,T)| < \alpha \min\{|S|,|T|\}$. Therefore, the outcome of the algorithm is a subset $V' \subseteq V$ of vertices of G, with $n^*/32 \le V' \le n^*$, such that $G[V']$ is a $\hat{\alpha}$-expander, for $\hat{\alpha} = \Omega\left(\frac{\alpha^2}{d \cdot 2^{10L(G)}}\right)$, in time $\mathrm{poly}(n) \cdot \left(\frac{256d}{\hat{\alpha}}\right)^{L(G)}$. Recall that $L(G) \le O(\log(d/\alpha))$. Therefore, we get that $\hat{\alpha} = \Omega\left(\frac{\alpha^2}{d \cdot 2^{O(\log(d/\alpha))}}\right) \ge (\alpha/d)^{c'_x}$ for some constant c'_x, and the running time of the algorithm is $\mathrm{poly}(n) \cdot \left(\frac{d}{\alpha}\right)^{O(\log(d/\alpha))}$. It now remains to prove Claim 6.5.

Proof of Claim 6.5. We denote $|V(G')| = n'$. We let c be a large enough constant. We prove by induction on $L(G')$ that the claim is true, with the running time of the algorithm bounded by $n^c \cdot (256d/\beta)^{L(G')}$. The base of the recursion is when $L(G') = 0$, and so $n^*/4 \le n' \le 4n^*/3$. We apply Corollary 6.4 to graph G' with the parameter β. If the outcome of the corollary is a subset $V' \subseteq V(G')$ of vertices with $n'/4 \le |V'| \le 3n'/4$, such that $G'[V']$ is an $\Omega(\beta^2/d)$-expander, then we terminate the algorithm and return V'. Notice that in this case, we are guaranteed that $n^*/16 \le |V'| \le n^*$. Otherwise, the algorithm returns a balanced cut (S,T) in G', with $|E_{G'}(S,T)| < \beta \cdot \min\{|S|,|T|\}$. We then return this cut. The running time of the algorithm is $\mathrm{poly}(n)$.

We now assume that the theorem holds for all graphs G' with $L(G') < i$, for some integer $i > 0$, and prove it for a given graph G' with $L(G') = i$. Let $n' = |V(G')|$. The proof is somewhat similar to the proof of Corollary 6.4. Throughout the algorithm, we maintain a balanced cut (U',U'') of G', with $|U'| \ge |U''|$. Initially, we start with an arbitrary such balanced cut. Notice that $|E(U',U'')| \le |E(G')| \le n'd$. While $|E(U',U'')| \ge \beta n'/4$, we perform iterations (that we call phases for convenience, since each of them consists of a number of iterations). At the end of every phase, we either compute a subset $V' \subseteq V(G')$ of vertices of G', such that $n^*/32 \le |V'| \le n^*$, and $G'[V']$ is an $\hat{\beta}$-expander, in which case we terminate the algorithm and return V'; or we compute a new balanced cut (J',J'') in G', such that $|E(J',J'')| \le |E(U',U'')| - \frac{\beta n'}{32}$. If $|E(J',J'')| < \beta n'/4$, then we return this cut; it is easy to verify that $|E(J',J'')| < \beta \cdot \min\{|J'|,|J''|\}$. Otherwise, we replace (U',U'') with the new cut (J',J''), and continue to the next iteration. Since initially $|E(U',U'')| \le n'd$, and since $|E(U',U'')|$ decreases by at least $\frac{\beta n'}{32}$ in every phase, the number of phases is bounded by $\frac{32d}{\beta}$. We now proceed to describe a single phase.

An execution of a phase. We assume that we are given a balanced cut (U', U'') in G', with $|U'| \geq |U''|$, and $|E(U', U'')| \geq \beta n'/4$. Our goal is to either compute a subset V' of vertices of G' such that $n^*/32 \leq |V'| \leq n^*$ and $G'[V']$ is an $\hat{\beta}$-expander, or return another balanced cut (J', J'') in G', with $|E(J', J'')| \leq |E(U', U'')| - \frac{\beta n'}{32}$. Let $\beta' = \beta/32$. Over the course of the algorithm, we will maintain a set E' of edges that we remove from the graph, starting with $E' = \emptyset$, and a collection $\mathcal{G}$ of subgraphs of $G[U']$ (that will contain at most 4 such subgraphs). As each graph $H \in \mathcal{G}$ is a subgraph of $G[U']$, we are guaranteed that $|V(H)| \leq 3n'/4$, and so $L(H) \leq L(G') - 1$. We start with $\mathcal{H}$ containing a single graph, the graph $G'[U']$. We then iterate, while there is a graph $H \in \mathcal{H}$ with $|V(H)| > |U'|/2$.

In every iteration, we let $H \in \mathcal{H}$ be the unique graph with $|V(H)| > |U'|/2$. Notice that $|V(H)| \geq n'/4 \geq n^*/3$, since we have assumed that $L(G') > 0$ and so $n' \geq 4n^*/3$. We apply the algorithm from the induction hypothesis to H, with the parameter $\beta' = \beta/32$. If the outcome is a subset $V' \subseteq V(H)$ of vertices of G', such that $n^*/32 \leq |V'| \leq n^*$ and $H[V']$ is a $\hat{\beta}'$-expander, for $\hat{\beta}' \geq \Omega\left(\frac{(\beta')^2}{d \cdot 2^{10L(H)}}\right)$ then we terminate the algorithm and return V'. Notice that, since $L(H) \leq L(G) - 1$, and $\beta' = \beta/32$, we get that $\frac{(\beta')^2}{d \cdot 2^{10L(H)}} \geq \frac{\beta^2}{d \cdot 2^{10L(G')}}$, so $G'[V']$ is a $\hat{\beta}$-expander. Otherwise, the algorithm returns a balanced cut (S, T) of $V(H)$, such that $|E(S, T)| < \beta' \cdot \min\{|S|, |T|\}$. We add the edges of $E(S, T)$ to E', remove H from $\mathcal{H}$, and add $H[S]$ and $H[T]$ to $\mathcal{H}$. The algorithm terminates once for every graph $H \in \mathcal{H}$, $|V(H)| \leq |U'|/2$. Let $r = |\mathcal{H}|$ at the end of the algorithm. Since the cuts (S, T) that we compute in every iteration are balanced, it is easy to verify that we run the algorithm from the induction hypothesis at most 3 times, and that $r \leq 4$, since in every iteration the size of the largest graph in $\mathcal{H}$ decreases by at least factor 3/4, and $(3/4)^3 < 1/2$. Denote $\mathcal{H} = \{H_1, \ldots, H_r\}$, and for each $1 \leq j \leq r$, let $V_j = V(H_j)$, and let $m_j = |E(V_j, U'')|$. Since $|E(U', U'')| \geq \beta n'/4$, there is some index $1 \leq j \leq r$, such that $|E(V_j, U'')| \geq \beta n'/16$. We define a new balanced cut (J', J''), by setting $J' = U' \setminus V_j$ and $J'' = U'' \cup V_j$. Since $|V_j| \leq |U'|/2$, it is immediate to verify that it is a balanced cut. Moreover, it is immediate to verify that $|E'| \leq \beta'|U'| \leq 3\beta' n'/4 \leq \beta n'/32$, and so:

$$|E(J', J'')| \leq |E(U', U'')| - |E(V_j, U'')| + |E'| \leq |E(U', U'')| - \frac{\beta n'}{16} + \frac{\beta n'}{32} \leq |E(U', U'')| - \frac{\beta n'}{32}.$$

Finally, we bound the running time of the algorithm. The running time is at most $\text{poly}(n)$ plus the time required for the recursive calls to the same procedure. Recall that the number of phases in the algorithm is at most $32d/\beta$, and every phase requires up to 3 recursive calls. Therefore, the total number of recursive calls is bounded by $100d/\beta$. Each recursive call is to a graph H that has $L(H) < L(G)$. From the induction hypothesis, the running time of each recursive call is bounded by $n^c \cdot \left(256d/\hat{\beta}'\right)^{L(G)-1} \leq n^c \cdot \left(256d/\hat{\beta}\right)^{L(G)-1}$, and so the total running time of the algorithm is bounded by:

$$n^c + \frac{100d}{\beta} \cdot n^c \cdot \left(\frac{256d}{\hat{\beta}}\right)^{L(G)-1} \leq n^c \cdot \left(\frac{256d}{\hat{\beta}}\right)^{L(G)},$$

since $\beta > \hat{\beta}$. □

□

We are now ready to complete the proof of Theorem 6.1.

Proof of Theorem 6.1. We start with the input n-vertex α-expander G and apply Lemma 6.2 to it, obtaining a subset $V_1 \subseteq V(G)$ of vertices, such that $G[V_1]$ is a $\hat{\alpha}^*$-expander and $\frac{\alpha n}{256d} \le |V_1| \le \frac{\alpha n}{8d}$. Let $E' = \delta_G(V_1)$. Since the maximum vertex degree in G is at most d, $|E'| \le \frac{\alpha n}{8}$.

We use the following claim, which is similar to Claim 2.3, except that it provides an efficient algorithm instead of the existential result of Claim 2.3, at the expense of obtaining somewhat weaker parameters. The proof appears in the Appendix.

Claim 6.6 *There is an efficient algorithm, that given an α-expander $G = (V, E)$ with maximum vertex degree at most d and a subset $E' \subseteq E$ of its edges, computes a subgraph $H \subseteq G \setminus E'$ that is an $\Omega\left(\frac{\alpha^2}{d}\right)$-expander, and $|V(H)| \ge |V| - \frac{4|E'|}{\alpha}$.*

We apply Claim 6.6 to graph G and the set E' of edges computed above. Let $H \subseteq G \setminus E'$ be the resulting graph, and let $V_2 = V(H)$. From Claim 6.6, $|V_2| \ge n - \frac{4|E'|}{\alpha} \ge n/2$. Since $|V_1| < n/2$ and the set E' of edges disconnects the vertices of V_1 from the rest of the graph, while H is an $\Omega\left(\frac{\alpha^2}{d}\right)$-expander and therefore a connected graph, $V_1 \cap V_2 = \emptyset$.

We are now ready to define the final partition (V', V'') of $V(G)$, by letting it be the minimum cut separating the vertices of V_1 from the vertices of V_2 in G: that is, we require that $V_1 \subseteq V'$, $V_2 \subseteq V''$, and among all such partitions (V', V'') of $V(G)$, we select the one minimizing $|E(V', V'')|$. The partition (V', V'') can be computed efficiently using standard techniques: we construct a new graph $\hat{G}$ by starting with G, contracting all vertices of V_1 into a source s, contracting all vertices of V_2 into a destination t, and computing a minimum s-t cut in the resulting graph. The resulting cut naturally defines the partition (V', V'') of $V(G)$. Let $E'' = E(V', V'')$, and denote $|E''| = z$. From Menger's theorem, there is a set $\mathcal{P}$ of z edge-disjoint paths in G, connecting V_1 to V_2. Therefore, there is a set $\mathcal{P}_1$ of z edge-disjoint paths in $G[V'] \cup E''$, where each path in $\mathcal{P}_1$ connects a distinct edge of E'' to a vertex of V_1, and similarly, there is a set $\mathcal{P}_2$ of z edge-disjoint paths in $G[V''] \cup E''$, where each path in $\mathcal{P}_2$ connects a distinct edge of E'' to a vertex of V_2.

We claim that each of the graphs $G[V'], G[V'']$ is an α^*-expander, for $\alpha^* = \frac{\alpha\hat{\alpha}^*}{512d}$. We prove this for $G[V']$; the proof for $G[V'']$ is similar. Assume for contradiction that $G[V']$ is not an α^*-expander. Then there is a cut (X, Y) in $G[V']$, such that $|E(X, Y)| < \alpha^* \cdot \min\{|X|, |Y|\}$. Assume w.l.o.g. that $|X \cap V_1| \le |Y \cap V_1|$. We now consider two cases.

The first case happens when $|X \cap V_1| \ge \frac{\alpha|X|}{512d}$. In that case, since $G[V_1]$ is an $\hat{\alpha}^*$-expander, there are at least $\hat{\alpha}^* \cdot |X \cap V_1| \ge \frac{\hat{\alpha}^* \cdot \alpha|X|}{512d} \ge \alpha^*|X|$ edges connecting $X \cap V_1$ to $Y \cap V_1$, and so $|E(X, Y)| > \alpha^* \cdot \min\{|X|, |Y|\}$, a contradiction. Therefore, we assume now that $|X \cap V_1| < \frac{\alpha|X|}{512d}$.

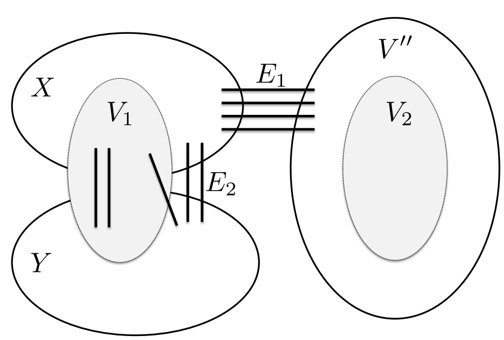


Figure 8: An illustration for the proof of Theorem 6.1

We partition the edges of $\delta_G(X)$ into two subsets: set E_1 contains all edges that lie in $E(V', V'')$, and set E_2 contains all remaining edges, so $E_2 = E(X, Y)$ (see Figure 8). Note that from the definition of the cut (X,Y), $|E_2|<\alpha^*|X|$. Recall that for every edge $e\in E(V',V'')$, there is a path $P_e\in\mathcal{P}_1$ contained in $G[V']\cup E(V',V'')$, connecting e to a vertex of V_1, such that all paths in $\mathcal{P}_1$ are edge-disjoint. Let $\tilde{\mathcal{P}}\subseteq\mathcal{P}_1$ be the set of paths originating at the edges of E_1. We further partition $\tilde{\mathcal{P}}$ into two subsets: set $\tilde{\mathcal{P}}'$ contains all paths P_e that contain an edge of E_2, and $\tilde{\mathcal{P}}''$ contains all remaining paths. Notice that $|\tilde{\mathcal{P}}'|\leq|E_2|<\alpha^*|X|$. On the other hand, every path $P_e\in\tilde{\mathcal{P}}''$ is contained in $G[X]\cup E_1$, and contains a vertex of $V_1\cap X$ – the endpoint of P_e. Since we have assumed that $|V_1\cap X|<\frac{\alpha|X|}{512d}$, and since the maximum vertex degree in G is at most d, while the paths in $\tilde{\mathcal{P}}''$ are edge-disjoint, we get that $|\tilde{\mathcal{P}}''|<\frac{\alpha|X|}{512}$. Altogether, we get that $|E_1|=|\tilde{\mathcal{P}}|\leq\alpha^*|X|+\frac{\alpha|X|}{512}$, and $|\delta_G(X)|=|E_1|+|E_2|\leq 2\alpha^*|X|+\frac{\alpha|X|}{512}\leq\frac{\alpha|X|}{256}<\alpha\cdot\min\{|X|,n/256\}\leq\alpha\cdot\min\{|X|,|V(G)\setminus X|\}$, since $|V(G)\setminus X|\geq n/256$, as $V_2\cap X=\emptyset$. This contradicts the fact that G is an α-expander. □

Corollary 6.7 *There is an algorithm, that, given, an n-vertex α-expander G with maximum vertex degree at most d and an integer $\ell\geq 1$, where $0<\alpha<1/3$, computes an α_ℓ-expanding Path-of-Sets system Σ of length ℓ and width $w_\ell=\lceil\alpha_\ell n\rceil$, together with a subgraph G_Σ of G, where $\alpha_\ell=\alpha^{c_x{}^{\ell-1}}/d^{c_x^{2\ell-2}}$, and $c_x\geq 3$ is the constant from Theorem 6.1. The running time of the algorithm is* $\text{poly}(n)\cdot\left(\frac{d}{\alpha_\ell}\right)^{O(\log(d/\alpha_\ell))}$.

We note that we will use the corollary for with $\ell=48$, and so the resulting Path-of-Sets System will have expansion $(\alpha/d)^{O(1)}$, and the running time of the algorithm from Corollary 6.7 is $\text{poly}(n)\cdot\left(\frac{d}{\alpha}\right)^{O(\log(d/\alpha))}$.

Proof: The proof is by induction on ℓ. The base case is when $\ell=1$. We choose two arbitrary disjoint subsets A_1,B_1 of $\lceil w_1\rceil<n/2$ of vertices, and we let $S_1=G$. This defines an α-expanding Path-of-Sets System of length 1 and width w_1.

We now assume that we are given an integer $\ell>1$, and an $\alpha_{\ell-1}$-expanding Path-of-Sets System $\Sigma=(\mathcal{S},\mathcal{M},A_1,B_{\ell-1})$ of length $\ell-1$ and width $w_{\ell-1}$, where $G_\Sigma\subseteq G$. We assume that $\mathcal{S}=(S_1,\ldots,S_{\ell-1})$. We compute an α_ℓ-expanding Path-of-Sets System $\Sigma'=(\mathcal{S}',\mathcal{M}',A'_1,B'_\ell)$ of length ℓ and width w_ℓ. We will denote $\mathcal{S}'=(S'_1,\ldots,S'_\ell)$, and for each $1\leq i\leq\ell'$, the corresponding vertex sets A_i and B_i in S'_i are denoted by A'_i and B'_i, respectively.

For all $1\leq i<\ell-1$, we set $S'_i=S_i$. We also let $A'_1\subseteq A_1$ be any subset of w_ℓ vertices, and for $1\leq i<\ell-2$, we let $\mathcal{M}'_i\subseteq\mathcal{M}_i$ be any subset of w_ℓ edges; the endpoints of these edges lying in B_i and A_{i+1} are denoted by B'_i and A'_{i+1} respectively. It remains to define $S'_{\ell-1},S'_\ell$, the matchings $\mathcal{M}'_{\ell-2}$ and $\mathcal{M}'_{\ell-1}$ (that implicitly define the sets $B'_{\ell-2},A'_{\ell-1},B'_{\ell-1},A'_\ell$ of vertices), and the set B'_ℓ of vertices.

We apply Theorem 6.1 to graph $S_{\ell-1}$, and compute, in time $\text{poly}(n)\cdot\left(\frac{d}{\alpha_{\ell-1}}\right)^{O(\log(d/\alpha_{\ell-1}))}$ a partition (V',V'') of $V(S_{\ell-1})$, such that $|V'|,|V''|\geq\frac{\alpha_{\ell-1}|V(S_{\ell-1})|}{256d}$, and each graph $G[V'],G[V'']$ is an α^*-expander, for $\alpha^*\geq\left(\frac{\alpha_{\ell-1}}{d}\right)^{c_x}$.

One of the two subsets, say V', must contain at least half of the vertices of $A_{\ell-1}$. We set $S'_{\ell-1}=S_{\ell-1}[V']$ and $S'_\ell=S_{\ell-1}[V'']$. Recall that: $|V'|,|V''|\geq\frac{\alpha_{\ell-1}|V(S_{\ell-1})|}{256d}\geq\frac{\alpha_{\ell-1}w_{\ell-1}}{128d}$. Since graph $S_{\ell-1}$ is an $\alpha_{\ell-1}$-expander, there are at least $\frac{\alpha^2_{\ell-1}w_{\ell-1}}{128d}$ edges connecting V' to V''. Since maximum vertex degree in G is at most d, there is a matching $\mathcal{M}$, between vertices of V' and vertices of V'', with $|\mathcal{M}|\geq\frac{\alpha^2_{\ell-1}w_{\ell-1}}{128d^2}$. We claim that $|\mathcal{M}|\geq w_\ell$. In order to see this, it is enough to prove that $w_\ell\leq\frac{\alpha^2_{\ell-1}w_{\ell-1}}{128d^2}$. Since $w_\ell=\lceil\alpha_\ell n\rceil$, this is equivalent to proving that:

$$\alpha_\ell \leq \frac{\alpha_{\ell-1}^3}{256d^2}.$$

This is easy to verify from the definition of α_ℓ and the fact that $c_x \geq 3$. We let $\mathcal{M}'_{\ell-1}$ be any subset of $\mathcal{M}$ containing w_ℓ edges. The endpoints of the edges of $\mathcal{M}'_{\ell-1}$ lying in V' and V'' are denoted by $B'_{\ell-1}$ and A'_ℓ respectively. We let B'_ℓ be any subset of w_ℓ vertices of $V'' \setminus A'_\ell$. Finally, we let $A'_{\ell-1}$ any subset of w_ℓ vertices of $(V' \cap A_{\ell-1}) \setminus B'_{\ell-1}$; $\mathcal{M}'_{\ell-2} \subseteq \mathcal{M}_{\ell-2}$ the subset of edges whose endpoints lie in $A'_{\ell-1}$; and $B'_{\ell-2}$ the set of endpoints of the edges of $\mathcal{M}'_{\ell-2}$ lying in $B_{\ell-2}$. This completes the construction of the Path-of-Sets System Σ'. It is immediate to verify that it has length ℓ, width w_ℓ, and that $G_{\Sigma'} \subseteq G$. It remains to prove that it is α_ℓ-expanding, or equivalently, that $S'_{\ell-1}$ and S'_ℓ are α_ℓ-expanders. Recall that Theorem 6.1 guarantees that both these graphs are α^*-expanders, where $\alpha^* \geq \left(\frac{\alpha_{\ell-1}}{d}\right)^{c_x}$. It is now enough to verify that $\alpha^* \geq \alpha_\ell$, which is immediate to do from the definition of α_ℓ:

$$\alpha^* \geq \left(\frac{\alpha_{\ell-1}}{d}\right)^{c_x} = \frac{\left(\alpha^{c_x{}^{\ell-2}}/d^{c_x^{2\ell-4}}\right)^{c_x}}{d^{c_x}} = \frac{\alpha^{c_x{}^{\ell-1}}}{d^{c_x^{2\ell-3}} \cdot d^{c_x}} \geq \frac{\alpha^{c_x{}^{\ell-1}}}{d^{c_x^{2\ell-2}}} = \alpha_\ell$$

Lastly, the running time of the algorithm is dominated by partitioning $S_{\ell-1}$, and is bounded by $\mathrm{poly}(n) \cdot \left(\frac{d}{\alpha_{\ell-1}}\right)^{O(\log(d/\alpha_{\ell-1}))} \leq \mathrm{poly}(n) \cdot \left(\frac{d}{\alpha_\ell}\right)^{O(\log(d/\alpha_\ell))}$, as required. □

We apply Corollary 6.7 to the input graph G, with the parameter $\ell = 48$, obtaining a sub-graph $G_\Sigma \subseteq G$, and an α'-expanding Path-of-Sets System Σ of length 48 and width $w' = \lceil \alpha' n \rceil$, where $\alpha' = (\alpha/d)^{O(1)}$. The running time of the algorithm is $\mathrm{poly}(n) \cdot \left(\frac{d}{\alpha}\right)^{O(\log(d/\alpha))}$.

6.2 Part 2: From Expanding to Strong Path-of-Sets System

The goal of this subsection is to prove the following theorem:

Theorem 6.8 *There is an efficient algorithm, that, given a parameter $\ell > 0$, and an α-Expanding Path-of-Sets System Σ of width w and length 4ℓ, where $0 < \alpha < 1$, such that the corresponding graph G_Σ has maximum vertex-degree at most d, computes a Strong Path-of-Sets System Σ', of width $w' = \Omega(\alpha^3 w/d^4)$ and length ℓ, such that the maximum vertex degree in the corresponding graph $G_{\Sigma'}$ is at most d, and $G_{\Sigma'}$ is a minor of G_Σ. Moreover, the algorithm computes a model of $G_{\Sigma'}$ in G_Σ.*

We use the following simple claim, whose proof is deferred to the Appendix.

Claim 6.9 *There is an efficient algorithm, that, given an α-expander G, whose maximum vertex degree is at most d, where $0 < \alpha < 1$, together with two disjoint subsets A, B of its vertices of cardinality z each, computes a collection $\mathcal{P}$ of $\lceil \alpha z/d \rceil$ disjoint paths, connecting vertices of A to vertices of B in G.*

We will also use the following theorem, whose proof is similar to some arguments that appeared in [CC16], and is deferred to the Appendix.

Theorem 6.10 *There is an efficient algorithm, that, given an α-Expanding Path-of-Sets System $\Sigma = (\mathcal{S}, \mathcal{M}, A_1, B_3)$ of width w and length 3, where $0 < \alpha < 1$, and the corresponding graph G_Σ has maximum vertex degree at most d, computes subsets $\hat{A}_1 \subseteq A_1, \hat{B}_3 \subseteq B_3$ of $\Omega(\alpha^2 w/d^3)$ vertices each, such that $\hat{A}_1 \cup \hat{B}_3$ is well-linked in G_Σ.*

We are now ready to complete the proof of Theorem 6.8.

Proof of Theorem 6.8. We construct a Strong Path-of-Sets System $\Sigma' = (\mathcal{S}', \mathcal{M}', A'_1, B'_\ell)$ of length ℓ and width w', denoting $\mathcal{S}' = (S'_1, \ldots, S'_\ell)$. For all $1 \leq i \leq \ell$, the corresponding vertex sets A_i and B_i are denoted by A'_i and B'_i, respectively.

For all $1 \leq i \leq \ell$, we let Σ_i be the α-expanding Path-of-Sets System of width w and length 3 obtained by using the clusters $S_{4i-3}, S_{4i-2}, S_{4i-1}$, and the matchings $\mathcal{M}_{4i-3}$ and $\mathcal{M}_{4i-2}$. In order to define the new Path-of-Sets System, for each $1 \leq i \leq \ell$, we set $S'_i = G_{\Sigma_i}$. We apply Theorem 6.10 to Σ_i, to obtain subsets $\hat{A}_i \subseteq A_{4i-3}$, $\hat{B}_i \subseteq B_{4i-1}$ of $\Omega(\alpha^2 w/d^3)$ vertices each, such that $\hat{A}_i \cup \hat{B}_i$ are well-linked in S'_i.

In order to complete the construction of the Path-of-Sets System Σ', we let $A'_1 \subseteq \hat{A}_1$ be any subset of w' vertices, and we define $B'_\ell \subseteq \hat{B}_\ell$ similarly. It remains to define, for each $1 \leq i < \ell$, the matching $\mathcal{M}'_i$. We will ensure that the endpoints of the resulting matching are contained in $\hat{B}_i$ and $\hat{A}_{i+1}$, respectively, ensuring that the resulting Path-of-Sets System is strong.

Consider some index $1 \leq i < \ell$. Recall that we have computed the sets $\hat{B}_i \subseteq B_{4i-1}$, $\hat{A}_{i+1} \subseteq A_{4i+1}$ of vertices. We let $E'_i \subseteq \mathcal{M}_{4i-1}$ be the set of edges incident to the vertices of $\hat{B}_i$, and we denote by $\tilde{A}_{4i} \subseteq A_{4i}$ the set of vertices in A_{4i} that serve as their endpoints. Similarly, we let $E''_i \subseteq \mathcal{M}_{4i}$ be the set of edges incident to the vertices of $\hat{A}_{i+1}$, and we denote by $\tilde{B}_{4i} \subseteq B_{4i}$ the set of vertices in B_{4i} that serve as their endpoints. From Claim 6.9, there is a set $\mathcal{Q}_i$ of disjoint paths in S_{4i}, connecting vertices of $\tilde{A}_{4i}$ to vertices of $\tilde{B}_{4i}$, of cardinality $w' = \Omega(\alpha^3 w/d^4)$. By extending the paths in $\mathcal{Q}_i$ to include the edges of $E'_i \cup E''_i$ incident to them, we obtain a collection $\mathcal{Q}'_i$ of w' disjoint paths in $S_{4i} \cup \mathcal{M}_{4i-1} \cup \mathcal{M}_{4i}$, connecting vertices of $\hat{B}_i$ to vertices of $\hat{A}_{i+1}$. We denote the endpoints of the paths in $\mathcal{Q}'_i$ lying in $\hat{B}_i$ by B'_i, and the endpoints of the paths in $\mathcal{Q}'_i$ lying in $\hat{A}_{i+1}$ by A'_{i+1}. The paths in $\mathcal{Q}'_i$ naturally define the matching $\mathcal{M}'_i$ between the vertices of B'_i and the vertices of A'_{i+1}. This concludes the definition of the Path-of-Sets System Σ'. It is immediate to verify that it is a strong Path-of-Sets System of length ℓ and width w', and to obtain a model of $G_{\Sigma'}$ in G_Σ. Note that graph $G_{\Sigma'}$ has maximum vertex degree at most d. $\square$

Recall that in Part 1 of the algorithm, we have obtained a sub-graph $G_\Sigma \subseteq G$, and an α'-expanding Path-of-Sets System Σ of length 48 and width $w' = \lceil \alpha' n \rceil$, where $\alpha' = (\alpha/d)^{O(1)}$. Applying Theorem 6.8 to Σ, we obtain a Strong Path-of-Sets System Σ' of length 12 and width $w'' = \Omega\left(\frac{(\alpha')^3 w'}{d^4}\right) = \Omega\left(\frac{(\alpha')^4}{d^4} n\right) = \left(\frac{\alpha}{d}\right)^{O(1)} \cdot n$. We have also computed a model of $G_{\Sigma'}$ in G, and established that the maximum vertex degree in $G_{\Sigma'}$ is at most d. For convenience, we let c' be a constant, such that $w'' \geq \frac{\alpha^{c'}}{d^{c'}} n$.

6.3 From Strong Path-of-Sets System to Path-of-Expanders System

The goal of this subsection is to prove the following theorem:

Theorem 6.11 *There is an efficient algorithm, that, given a Strong Path-of-Sets System Σ of width w and length 12, such that the corresponding graph G_Σ has at most n vertices and has maximum vertex degree at most d, computes a Path-of-Expanders System Π of width $\hat{w} = \Omega\left(\frac{w^4}{d^2 n^3}\right)$ and expansion $\hat{\alpha} \geq \Omega\left(\frac{w^2}{n^2 d}\right)$, whose corresponding graph G_Π has maximum vertex degree at most $d+1$ and is a minor of G_Σ. Moreover, the algorithm computes a model of G_Π in G_Σ.*

Before we prove Theorem 6.11, we complete the proof of Theorem 2.4 using it. Recall that our input is an α-expander G, for some $0 < \alpha < 1$, with $|V(G)| = n$, such that the maximum vertex degree in

G is at most d. Our goal is to provide an algorithm that computes a Path-of-Expanders System Π of expansion $\tilde{\alpha} \geq \left(\frac{\alpha}{d}\right)^{\hat{c}_1}$ and width $\tilde{w} \geq n \cdot \left(\frac{\alpha}{d}\right)^{\hat{c}_2}$, such that the maximum vertex degree in G_Π is at most $d+1$, and to compute a minor of G_Π in G.

Recall that in Step 2 we have constructed a Strong Path-of-Sets System Σ' of length 12 and width $w'' \geq \frac{\alpha^{c'}}{d^{c'}} n$, for some constant c', such that $G_{\Sigma'}$ has maximum vertex degree at most d. We have also computed a model of $G_{\Sigma'}$ in G. Our last step is to apply Theorem 6.11 to Σ'. As a result, we obtain a Path-of-Expanders System Π of width $\hat{w} = \Omega\left(\frac{(w'')^4}{d^2 n^3}\right)$ and expansion $\hat{\alpha} \geq \Omega\left(\frac{(w'')^2}{n^2 d}\right)$, whose corresponding graph G_Π has maximum vertex degree at most $d+1$. We also obtain a model of G_Π in G_Σ.

Substituting the value $w'' \geq \frac{\alpha^{c'}}{d^{c'}} n$, we get that the width of the Path-of-Expanders System is $\Omega\left(\frac{\alpha^{4c'}}{d^{2+4c'}}\right) \cdot n$, and that its expansion is $\Omega\left(\frac{\alpha^{2c'}}{d^{2c'+1}}\right)$. By appropriately setting the constants $\hat{c}_1$ and $\hat{c}_2$, we ensure that the width of the Path-of-Expanders System is at least $n \cdot \left(\frac{\alpha}{d}\right)^{\hat{c}_2}$ and its expansion is at least $\left(\frac{\alpha}{d}\right)^{\hat{c}_1}$.

In the remainder of this section, we prove Theorem 6.11. We can assume w.l.o.g. that $w^4 \geq 2^{14} n^3 d^2$, since otherwise it is sufficient to produce a Path-of-Expanders System of width 1, which is trivial to do. We denote the input Strong Path-of-Sets System by $\Sigma = (\mathcal{S}, \mathcal{M}, A_1, B_{12})$, where $\mathcal{S} = (S_1, \ldots, S_{12})$, and we let G_Σ be its corresponding graph. For convenience, we denote by $\mathcal{I}_{\text{even}}$ and $\mathcal{I}_{\text{odd}}$ the sets of all even and all odd indices in $\{1, \ldots, 12\}$, respectively. The algorithm consists of three steps. In the first step, for every index $i \in \mathcal{I}_{\text{even}}$, we find a large set $\mathcal{P}_i$ of disjoint paths connecting A_i to B_i in S_i, and a subgraph $T_i \subseteq S_i$ that is an $\hat{\alpha}$-expander, such that the paths in $\mathcal{P}_i$ are disjoint from T_i. In the second step, for each such index $i \in \mathcal{I}_{\text{even}}$, we compute another set $\mathcal{Q}_i$ of disjoint paths in S_i, and a large enough subset $\mathcal{P}'_i \subseteq \mathcal{P}_i$ of paths, such that every path in $\mathcal{Q}_i$ connects a vertex on a distinct path of $\mathcal{P}'_i$ to a distinct vertex of T_i. In the third and the final step we compute the Path-of-Expanders System Π and a model of G_Π in G_Σ.

Step 1. In this step, we prove the following lemma.

Lemma 6.12 *There is an efficient algorithm, that, given an index $i \in \mathcal{I}_{even}$, computes a set $\mathcal{P}_i$ of $\left\lfloor \frac{w^2}{16nd} \right\rfloor$ paths in S_i, and a subgraph $T_i \subseteq S_i$, such that:*

- *graph T_i is an $\hat{\alpha}$-expander, and it contains at least $w/2$ vertices of A_i;*
- *the paths in $\mathcal{P}_i$ are disjoint from each other; they are also disjoint from T_i and internally disjoint from $A_i \cup B_i$;*
- *every path in $\mathcal{P}_i$ connects a vertex of A_i to a vertex of B_i; and*
- *every path in $\mathcal{P}_i$ has length at most $2n/w$.*

Proof: For convenience, we omit the subscript i in this proof. We are given a graph S that contains at most n vertices and has maximum vertex degree at most d, and two disjoint subsets A, B of $V(S)$ of cardinality w each, such that each of $A \cup B$ is well-linked in S. Therefore, there is a set $\mathcal{P}$ of w disjoint paths in S, connecting vertices of A to vertices of B, such that the paths in $\mathcal{P}$ are internally disjoint from $A \cup B$. We say that a path in $\mathcal{P}$ is *short* if it contains at most $2n/w$ vertices, and otherwise it is long. Since $|V(S)| \leq n$, at most $w/2$ paths in $\mathcal{P}$ can be long, and the remaining paths must be short. Let $\mathcal{P}' \subseteq \mathcal{P}$ be any subset of $\left\lfloor \frac{w^2}{16nd} \right\rfloor$ paths in $\mathcal{P}$. It is now sufficient to show an algorithm that computes an $\hat{\alpha}$-expander $T \subseteq S$, such that T is disjoint from the paths in $\mathcal{P}'$. In order to do so, we let E' be the set of all edges lying on the paths in $\mathcal{P}'$, so $|E'| \leq |\mathcal{P}'| \cdot \frac{2n}{w} \leq \left\lfloor \frac{w^2}{16nd} \right\rfloor \cdot \frac{2n}{w} \leq \frac{w}{8}$.

We start with $T = S \setminus E'$, and then iteratively remove edges from T, until we obtain a connected component of the resulting graph that is an $\hat{\alpha}$-expander, containing at least $w/2$ vertices of A. Notice that the original graph T is not necessarily connected. We also maintain a set E'' of edges that we remove from T, initialized to $E'' = \emptyset$. Our algorithm is iterative. In every iteration, we apply Theorem 2.2 to the current graph T, to obtain a cut (Z, Z') in T. If the sparsity of the cut is at least $\frac{w}{16n}$, that is, $|E_T(Z, Z')| \geq \frac{w}{16n} \min\{|Z|, |Z'|\}$, then we terminate the algorithm. Theorem 2.2 then guarantees that the expansion of T is $\Omega\left(\frac{w^2}{n^2 d}\right)$, that is, T is a $\hat{\alpha}$-expander. Otherwise, $|E_T(Z, Z')| < \frac{w}{16n} \min\{|Z|, |Z'|\}$. Assume w.l.o.g. that $|Z \cap A| \geq |Z' \cap A|$. We then add the edges of $E_T(Z, Z')$ to E'', set $T = T[Z]$, and continue to the next iteration. Note that the number of edges added to E'' during this iteration is at most $\frac{|Z'|w}{16n}$.

Clearly, the graph T we obtain at the end of the algorithm is an $\hat{\alpha}$-expander, and it is disjoint from all paths in $\mathcal{P}'$. It now only remans to show that T contains at least $w/2$ vertices of A. Assume for contradiction that this is false.

Assume that the algorithm performs r iterations, and for each $1 \leq j \leq r$, let (Z_j, Z'_j) be the cut computed by the algorithm in iteration j, where $|Z_j \cap A| \geq |Z'_j \cap A|$. But then for all $1 \leq j \leq r$, $|Z'_j \cap A| \leq w/2$ must hold. Let $n_j = |Z'_j \cap A|$. Since the vertices of A are well-linked in S, $\delta_S(Z'_j) \geq n_j$. Therefore:

$$\sum_{j=1}^{r} |\delta_S(Z'_j)| \geq \sum_{j=1}^{r} n_j \geq w/2,$$

since we have assumed that the final graph T has fewer than $w/2$ vertices of A. On the other hand, all edges in $\bigcup_{j=1}^{r} \delta_S(Z'_j)$ are contained in $E' \cup E''$, and so:

$$\sum_{j=1}^{r} |\delta_S(Z'_j)| \leq 2|E' \cup E''|.$$

Recall that $|E'| \leq \frac{w}{8}$, and it is easy to verify that $|E''| \leq \frac{w}{16n} \cdot n = \frac{w}{16}$. Therefore, $\sum_{j=1}^{r} |\delta_S(Z'_j)| < \frac{w}{2}$, a contradiction. □

Step 2. For every index $i \in \mathcal{I}_{\text{even}}$, let $A'_i \subseteq A_i$ be the subset of vertices that serve as endpoints for the paths in $\mathcal{P}_i$. The goal of this step is to prove the following lemma.

Lemma 6.13 *There is an efficient algorithm, that, given an index $i \in \mathcal{I}_{even}$, computes a subset $\mathcal{P}'_i \subseteq \mathcal{P}_i$ of $\hat{w}$ paths, and, for each path $P \in \mathcal{P}'_i$, a path Q_P in S_i, that connects a vertex of P to a vertex of T_i, such that the paths in set $\mathcal{Q}_i = \{Q_P \mid P \in \mathcal{P}'_i\}$ are disjoint from each other, internally disjoint from T_i, and internally disjoint from the paths in $\mathcal{P}'_i$.*

Proof: We fix an index $i \in \mathcal{I}_{\text{even}}$, and for convenience omit the subscript i for the remainder of the proof. Recall that we are given a set $A' \subseteq A$ of $\left\lfloor \frac{w^2}{16nd} \right\rfloor$ vertices, that serve as endpoints of the paths in $\mathcal{P}$. Recall that T contains at least $w/2$ vertices of A. We let $A'' \subseteq A$ be any set of $\left\lfloor \frac{w^2}{16nd} \right\rfloor$ vertices of A lying in T. Since the set A of vertices is well-linked in S, there is a set $\mathcal{Q}$ of $\left\lfloor \frac{w^2}{16nd} \right\rfloor$ node-disjoint paths, connecting the vertices of A' to the vertices of A'' in S. We say that a path in $\mathcal{Q}$ is *short* if it contains fewer than $\frac{64n^2d}{w^2}$ vertices, and otherwise we say that it is *long*. Since S contains at most n vertices, and the paths in $\mathcal{Q}$ are disjoint, at most $\frac{w^2}{64nd}$ paths of $\mathcal{Q}$ are long. We let $\hat{\mathcal{Q}} \subseteq \mathcal{Q}$ be the set of all short paths, so $|\hat{\mathcal{Q}}| \geq \frac{w^2}{64nd}$, and we let $\hat{A} \subseteq A'$ be the set of vertices that serve as endpoints of the paths in $\hat{\mathcal{Q}}$. We also let $\hat{\mathcal{P}} \subseteq \mathcal{P}$ the set of paths originating from the vertices in $\hat{A}$. We are now ready to compute the set $\mathcal{P}'$ of paths, and the corresponding paths Q_P for all $P \in \mathcal{P}'$.

We start with $\mathcal{P}' = \emptyset$, and then iterate. While $\hat{\mathcal{P}} \neq \emptyset$, let P be any path in $\hat{\mathcal{P}}$, and let $a \in \hat{A}$ be the vertex from which it originates. Let Q be the path of $\hat{\mathcal{Q}}$ originating at a. We prune the path Q as needed, so that it connects a vertex of P to a vertex of T, but is internally disjoint from P and T. Let Q' be the resulting path. We then add P to $\mathcal{P}'$, and we let $Q_P = Q'$. Next, we delete from $\hat{\mathcal{P}}$ all paths that intersect Q' (since the length of Q' is at most $\frac{64n^2d}{w^2}$, we delete at most $\frac{64n^2d}{w^2}$ paths from $\hat{\mathcal{P}}$), and for every path P^* that we delete from $\hat{\mathcal{P}}$, we delete from $\hat{\mathcal{Q}}$ the path sharing an endpoint with P^* (so at most $\frac{64n^2d}{w^2}$ paths are deleted from $\hat{\mathcal{Q}}$). Similarly, we delete from $\hat{\mathcal{Q}}$ every path that intersects P (since the length of P is at most $2n/w$, we delete at most $\frac{2n}{w} \leq \frac{64n^2d}{w^2}$ paths from $\hat{\mathcal{Q}}$), and for every path Q^* that we delete from $\hat{\mathcal{Q}}$, we delete from $\hat{\mathcal{P}}$ the path sharing an endpoint with Q^* (again, at most $\frac{64n^2d}{w^2}$ paths are deleted from $\hat{\mathcal{P}}$). Overall, we delete at most $\frac{128n^2d}{w^2}$ paths from $\hat{\mathcal{P}}$, and at most $\frac{128n^2d}{w^2}$ paths from $\hat{\mathcal{Q}}$. The paths that remain in both sets form pairs – that is, for every path $P^* \in \hat{\mathcal{P}}$, there is a path $Q^* \in \hat{\mathcal{Q}}$ originating at the same vertex of A, and vice versa. Furthermore, and all paths in $\hat{\mathcal{P}} \cup \hat{\mathcal{Q}}$ are disjoint from the paths in $\mathcal{P}' \cup \{Q_P \mid P \in \mathcal{P}'\}$.

At the end of the algorithm, we obtain a subset $\mathcal{P}' \subseteq \mathcal{P}$ of paths, and for each path $P \in \mathcal{P}'$, a path Q_P in S, connecting a vertex of P to a vertex of T, such that the paths in set $\mathcal{Q}' = \{Q_P \mid P \in \mathcal{P}'\}$ are disjoint from each other, internally disjoint from T, and internally disjoint from the paths in $\mathcal{P}'$. It now only remains to show that $|\mathcal{P}'| \geq \hat{w}$.

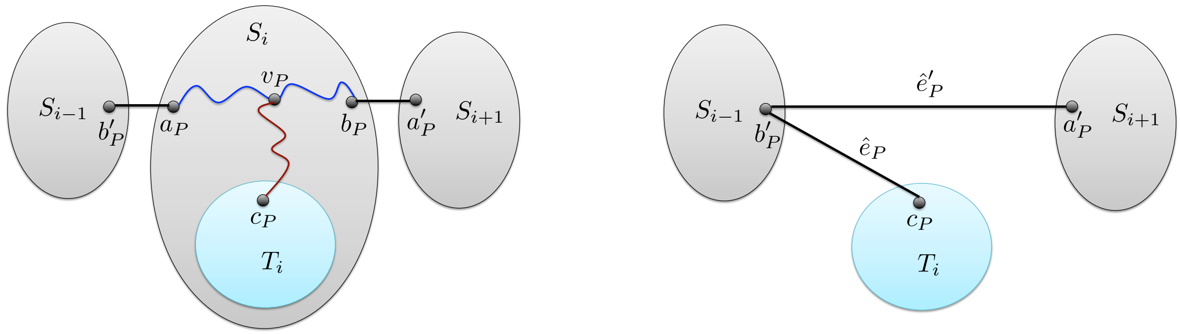
Recall that we start with $|\hat{\mathcal{P}}| \geq \frac{w^2}{64nd}$. In every iteration, we add one path to $\mathcal{P}'$, and delete at most $\frac{128n^2d}{w^2}$ paths from $\hat{\mathcal{P}}$. Since we have assumed that $w^4 \geq 2^{14}n^3d^2$, we get that $\frac{256n^2d}{w^2} \leq \frac{w^2}{64nd}$. It is then easy to verify that at the end of the algorithm, $|\mathcal{P}'| \geq \left\lfloor \frac{|\hat{\mathcal{P}}|}{256n^2d/w^2} \right\rfloor \geq \Omega\left(\frac{w^4}{n^3d^2}\right) = \hat{w}$. □

Step 3. In this step we complete the construction of the Path-of-Expanders System Π. We will also define a minor G' of G_Σ and compute a model of G_Π in G'; it is then easy to obtain a model of G_Π in G_Σ.

Consider some index $i \in \mathcal{I}_{\text{even}}$, and the sets $\mathcal{P}'_i, \mathcal{Q}_i$ of paths computed in Step 2. Let $P \in \mathcal{P}'_i$ be any such path, and assume that it connects a vertex $a_P \in A_i$ to a vertex $b_P \in B_i$. Let $v_P \in P$ be the endpoint of Q_P lying on P, and let c_P be its other endpoint. Finally, let e_P be the edge of $\mathcal{M}_{i-1}$ incident to a_P and let $b'_P \in B_{i-1}$ be its other endpoint. Similarly, if $i \neq 12$, let e'_P be the edge of $\mathcal{M}_i$ incident to b_P, and let $a'_P \in A_{i+1}$ be its other endpoint (see Figure 9(a)).

We contract the edge e_P and all edges lying on the sub-path of P between a_P and v_P, so that v_P and b'_P merge. The resulting vertex is denoted by b'_P. We also suppress all inner vertices on the path Q_P, obtaining an edge $\hat{e}_P$, connecting b'_P to c_P. Finally, if $i \neq 12$, then we contract all edges on the sub-path of P between v_P and b_P, obtaining an edge $\hat{e}'_P = (b_P, a'_P)$. We let $\hat{E}_i = \{\hat{e}_P \mid P \in \mathcal{P}'_i\}$ and we let $\hat{E}'_i = \{\hat{e}'_P \mid P \in \mathcal{P}'_i\}$ be the sets of these newly defined edges. Notice that the edges of $\hat{E}_i$ connect a subset of $\hat{w}$ vertices of B_{i-1} (that we denote by $\hat{B}_{i-1}$) to a subset of $\hat{w}$ vertices of T_i (that we denote by $\hat{C}_i$), and for $i \neq 12$, the edges of $\hat{E}'_i$ connect every vertex of $\hat{B}_{i-1}$ to some vertex of A_{i+1}; we denote the set of endpoints of these edges that lie in A_{i+1} by $\hat{A}_{i+1}$.

Once we perform this procedure for every path $P \in \mathcal{P}'_i$, for all $i \in \mathcal{I}_{\text{even}}$, we delete from the resulting



(a) Paths P (shown in blue) and Q_P (shown in red) before edge contractions

(b) After edge contractions

Figure 9: The contractions of the edges on paths P and Q_P.

graph all edges and vertices except those lying in graphs S_i for $i \in \mathcal{I}_{\text{odd}}$, graphs T_i for $i \in \mathcal{I}_{\text{even}}$, and the edges in $\hat{E}_i \cup \hat{E}'_i$ for $i \in \mathcal{I}_{\text{even}}$. The resulting graph, denoted by G', is a minor of G, and it is easy to verify that its maximum vertex degree is at most $d+1$.

We now define a Path-of-Expanders System $\Pi = (\tilde{\Sigma}, \tilde{\mathcal{M}}, \tilde{A}_1, \tilde{B}_6, \tilde{\mathcal{T}}, \tilde{\mathcal{M}}')$, where the clusters of $\tilde{\Sigma}$ are denoted by $\tilde{S}_1, \ldots, \tilde{S}_6$; for each $1 \le i \le 6$ the corresponding sets A_i, B_i, C_i of vertices are denoted by $\tilde{A}_i$, $\tilde{B}_i$ and $\tilde{C}_i$ respectively; the matching $\mathcal{M}'_i$ is denoted by $\tilde{\mathcal{M}}'_i$ and the expander T_i is denoted by $\tilde{T}_i$. For all $1 \le i < 6$, we also denote the matching $\mathcal{M}_i$ by $\tilde{\mathcal{M}}_i$.

For each $1 \le i \le 6$, we let the cluster $\tilde{S}_i$ of $\tilde{\Sigma}$ be S_{2i-1}, and we let the expander $\tilde{T}_i$ be T_{2i}. We also set $C_i = \hat{C}_{2i}$, and $\tilde{\mathcal{M}}'_i = \hat{E}_{2i}$. If $i > 1$, then we let $\tilde{A}_i = \hat{A}_{2i-1}$, and we let $\tilde{A}_1$ be any subset of $\hat{w}$ vertices of A_1. Similarly, if $i < 6$, then we let $\tilde{B}_i = \hat{B}_{2i-1}$, and we let $\tilde{B}_6$ be any subset of $\hat{w}$ vertices of B_6. Finally, for $i < 6$, we let $\tilde{\mathcal{M}}_i = \hat{E}'_{2i}$. It is immediate to verify that we have obtained a Path-of-Expanders System of width $\hat{w}$ and expansion $\hat{\alpha}$, and a model of G_Π in G'. It is now immediate to obtain a model of G_Π in G_Σ.

7 Proof of Theorem 1.2

The goal of this section is to provide the proof of Theorem 1.2. Notice that Theorem 1.2 provides slightly weaker dependence on n in the minor size than Theorem 1.1, but it has several advantages: its proof is much simpler, the algorithm's running time is polynomial in n, d and α, and it provides a better dependence on α and d in the bound on the minor size. Our algorithm also has an additional useful property: if it fails to find the required model, then with high probability it certifies that the input graph is not an α-expander by exhibiting a cut of sparsity less than α.

Let $G = (V, E)$ be the given n-vertex α-expander with maximum vertex degree at most d. As in the proof of Theorem 1.1, given a graph H with n' vertices and m' edges, we can construct another graph H', whose maximum vertex-degree is at most 3 and $|V(H')| \le n' + 2m' \le 2\left\lfloor \frac{n}{\tilde{c}^* \log^2 n} \cdot \frac{\alpha^3}{d^5} \right\rfloor$, such that H is a minor of H'. It is now enough to provide an efficient algorithm that computes a model of H' in G. For convenience of notation, we denote H' by $H = (U, F)$, and we denote $U = \left\{u_1, \ldots, u_{|U|}\right\}$. We can assume that $n > c_0$ for a large enough constant c_0 by appropriately setting the constant $\tilde{c}^*$, as otherwise it is enough to show that every graph of size 1 is a minor of G, which is trivial.

Our algorithm consists of a number of iterations. We say that a partition (V', V'') of V is *good* iff $|V'|, |V''| \ge n/(4d)$; and $G[V'], G[V'']$ are both connected graphs. We start with an arbitrary good partition (V_1, V_2) of V, obtained by using the algorithm from Observation 3.2 with $r = 2$. Assume without loss of generality that $|V_1| \geq |V_2|$. We now try to compute a model of H in G, by first embedding the vertices of H into connected sub-graphs of $G[V_2]$, and then routing the edges of H in $G[V_1]$. We show an efficient algorithm, that with high probability returns one of the following:

- either a good partition (V_1', V_2') such that $|E(V_1', V_2')| < |E(V_1, V_2)|$ (in this case, we proceed to the next iteration); or
- a model of H in G (in this case, we terminate the algorithm and return the model).

Clearly, we terminate after $|E|$ iterations, succeeding with high probability. We now describe a single iteration in detail. Recall that we are given a good partition (V_1, V_2) of V with $|V_1| \geq |V_2|$. Since G is an α-expander, we have $|E(V_1, V_2)| \geq \alpha n/(4d)$ (note that, if this is not the case, we have found a cut (V_1, V_2) of sparsity less than α). Since the maximum vertex-degree in G is bounded by d, we can efficiently find a matching $\mathcal{M} \subseteq E(V_1, V_2)$ of cardinality at least $\alpha n/(8d^2)$. We denote the endpoints of the edges in $\mathcal{M}$ lying in V_1 and V_2 by Z and Z', respectively. Let $\rho := 3 \cdot \lceil 4cd^2 \log^2 n/\alpha^2 \rceil$, where c is the constant from Lemma 5.1.

Recall that U is the set of vertices in the graph H. We apply Observation 3.2 to the graph $G[V_2]$, together with $R = Z'$ and parameter $r = |U|$, to obtain a collection $\mathcal{W} = \{W_1, \ldots, W_{|U|}\}$ of disjoint connected subgraphs of $G[V_2]$, such that for all $1 \leq i \leq |U|$,

$$|V(W_i) \cap Z'| \geq \left\lfloor \frac{|Z'|}{d|U|} \right\rfloor \geq \left\lfloor \frac{\alpha n}{8d^3|U|} \right\rfloor \geq \left\lfloor \frac{\alpha n}{8d^3} \cdot \frac{\tilde{c}^* d^5 \log^2 n}{2n\alpha^3} \right\rfloor = \left\lfloor \frac{\tilde{c}^* d^2 \log^2 n}{16\alpha^2} \right\rfloor$$

Here, we have used the fact that $|U| \leq 2 \left\lfloor \frac{n}{\tilde{c}^* \log^2 n} \cdot \frac{\alpha^3}{d^5} \right\rfloor$). By appropriately setting the constant $\tilde{c}^*$ in the bound on $|U|$, we can ensure that for all $1 \leq i \leq |U|$, $|V(W_i) \cap Z'| \geq 3\rho$.

Recall that we are given a graph $H = (U, F)$ with maximum vertex-degree 3 and that we have denoted $U = \{u_1, \ldots, u_{|U|}\}$. For $1 \leq i \leq |U|$, we think of the graph W_i as representing the vertex u_i of H. For each $1 \leq i \leq |U|$, and for each edge $e \in \delta_H(u_i)$, we select an arbitrary subset $Z_i'(e) \subseteq V(W_i) \cap Z'$ of ρ vertices, such that all resulting sets $\{Z_i'(e) \mid e \in \delta_H(u_i)\}$ of vertices are mutually disjoint. Let $E_i(e) \subseteq \mathcal{M}$ be the subset of edges of $\mathcal{M}$ that have an endpoint in $Z_i'(e)$, so $|E_i(e)| = \rho$. We let $Z_i(e)$ be the set of vertices of Z that serve as endpoints of the edges in $E_i(e)$. Notice that all resulting sets $\{Z_i(e) \mid 1 \leq i \leq |U|, e \in \delta_H(u_i)\}$ are mutually disjoint, and each of them contains ρ' vertices.

We apply the algorithm of Lemma 5.1 to the graph $G[V_1]$, together with the parameter $\alpha/2$ and the family $\{Z_i(e) \mid 1 \leq i \leq |U|, e \in \delta_H(u_i)\}$ of vertex subsets, that we order appropriately.

Case 1. The algorithm returns a cut. In this case, we obtain a cut (X, Y) in $G[V_1]$ of sparsity less than $\alpha/2$. We will compute a good partition (V_1', V_2') of V with $|E(V_1', V_2')| < |E(V_1, V_2)|$. We need the following simple observation whose proof appears in Appendix.

Observation 7.1 *There is an efficient algorithm, that given a connected graph $G = (V, E)$ and a cut (X, Y) in G, produces a cut (X^*, Y^*), whose sparsity is less than or equal to that of (X, Y), such that both $G[X^*]$ and $G[Y^*]$ are connected.*

We apply Observation 7.1 to graph $G[V_1]$ and cut (X, Y), obtaining a new cut (X^*, Y^*) of sparsity less than $\alpha/2$, such that both $G[X^*]$ and $G[Y^*]$ are connected. For convenience, we denote the cut (X^*, Y^*) by (X, Y), and we assume without loss of generality that $|Y| \leq |X|$. Notice that $|Y| \leq |V_1|/2 \leq |V|/2$.

Since G is an α-expander, $|\delta_G(Y)| \geq \alpha|Y|$ (note that, if this is not the case, then have found a cut $(Y, V\backslash Y)$ of sparsity less than α.).

Since $\delta_G(Y) = E(X,Y) \cup E(Y,V_2)$, we get that $|E(Y,V_2)| \geq \alpha|Y|/2$, and $|E(X,Y)| < |E(Y,V_2)|$. In particular, $E(Y,V_2) \neq \emptyset$. We now define a new cut (V_1', V_2') of G, where $V_2' = V_2 \cup Y$ and $V_1' = X$. We claim that (V_1', V_2') is a good partition of $V(G)$. It is immediate to verify that $|V_1'|, |V_2'| \geq n/(4d)$, and that $G[V_1'] = G[X]$ is connected. Moreover, since $G[Y]$ is connected and $E(Y,V_2) \neq \emptyset$, $G[V_2'] = G[V_2 \cup Y]$ is also connected. Lastly, we claim that $|E(V_1', V_2')| < |E(V_1, V_2)|$. Indeed, since $|E(X,Y)| < |E(V_2, Y)|$:

$$|E(V_1', V_2')| = |E(V_1, V_2)| - |E(V_2, Y)| + |E(Y, X)| < |E(V_1, V_2)|.$$

Therefore, we have computed a good partition (V_1', V_2') of $V(G)$, with $|E(V_1', V_2')| < |E(V_1, V_2)|$ as required.

Case 2. The algorithm returns paths. In this case, we have obtained, for every edge $e = (u_i, u_j) \in F$, a path $Q(e)$ in $G[V_1]$, connecting a vertex of $Z_i(e)$ to a vertex of $Z_j(e)$, such that, with high probability, the paths in $\{Q(e) \mid e \in F\}$ are mutually disjoint. If the paths in $\{Q(e) \mid e \in F\}$ are not mutually disjoint, the algorithm fails. We assume from now on that the paths in $\{Q(e) \mid e \in F\}$ are mutually disjoint. We extend each path $Q(e)$ to include the two edges of $\mathcal{M}$ that are incident to its endpoints, so that $Q(e)$ now connects a vertex of $Z_i'(e)$ to a vertex of $Z_j'(e)$.

We are now ready to define the model of H in G. For every $1 \leq i \leq |U|$, we let $f(u_i) = W_i$, and for every edge $e \in F$, we let $f(e) = Q(e)$. It is immediate to verify that this mapping indeed defines a valid model of H in G. This completes the proof of Theorem 1.2.

A Proof of Corollary 1.3.

In this subsection we prove Corollary 1.3. We use the following result of Krivelevich [Kri18b]:

Theorem A.1 (Corollary 1 of [Kri18b]) *For every $\epsilon > 0$, there exists $\gamma > 0$, such that for every $n > 0$, a random graph $G \sim \mathcal{G}(n, \frac{1+\epsilon}{n})$ contains an induced bounded-degree γ-expander $\tilde{G}$ on at least γn vertices w.h.p.*

Let $G \sim \mathcal{G}(n, \frac{1+\epsilon}{n})$. From the above theorem, w.h.p., there is an induced bounded-degree γ-expander $\tilde{G} \subseteq G$ on at least γn vertices, for some γ depending only on ϵ. From Theorem 1.1, every graph H of size at most $c_\epsilon n/\log n$ is a minor of $\tilde{G}$, where c_ϵ is some constant depending on ϵ only. Corollary 1.3 now follows. $\square$

B Proof of Observation 1.4.

Recall that we are given a integer s and a graph $G = (V,E)$ of size s. Assume for now that $2 \leq s < 2^{20}$. Let H_G be a graph with $s+1$ vertices and 0 edges. Notice that the number of vertices in H_G is strictly more than that in G, and hence H_G is not a minor of G. The observation now follows since $20s/\log s \geq s+1$. Thus from now on, we assume that $s \geq 2^{20}$ and hence, $20s/\log s \geq 2^{20}$.

We denote by $\mu(G) = |\{H \mid H \text{ is a minor of } G\}|$. For an integer r, let $\mathcal{F}_r$ be the set of all graphs of size at most r. The following two observations now complete the proof of Observation 1.4.

Observation B.1 $\mu(G) \le 3^s$.

Proof: From the definition of minors, every minor H of G can be identified by a subset $E_H^{\text{del}} \subseteq E$ of deleted edges, a subset $E_H^{\text{cont}} \subseteq E$ of contracted edges and a subset $V_H^{\text{del}} \subseteq V$ of deleted vertices. Thus,

$$\mu(G) \le 2^{|V|} \cdot 3^{|E|} \le 3^{|V|+|E|} \le 3^s.$$

□

Observation B.2 *For every even integer $r \ge 2^{10}$, $|\mathcal{F}_r| \ge r^{r/10}$.*

Proof: Let $k = \lfloor r^{0.9} \rfloor$. We lower-bound the number of graphs containing exactly k vertices and exactly $r/2$ edges. Notice that, since $r \ge 2^{10}$, $k + r/2 \le r$. For convenience, assume that the set $V^* = \{1, \ldots, k\}$ of vertices and their indices are fixed. We will first lower-bound the number of vertex-labeled graphs with the set V^* of vertices, that contain exactly $r/2$ edges. Since there are only $\binom{k}{2}$ 'edge-slots', this number is at least:

$$\binom{\binom{k}{2}}{r/2} \ge \binom{r^{1.6}}{r/2} \ge \left(\frac{r^{1.6} - r/2}{r/2}\right)^{r/2} \ge \left(r^{0.6}\right)^{r/2} \ge r^{0.3r}.$$

Here, the inequalities hold for all $r \ge 2^{10}$. Notice that two graphs $G_1 = (V^*, E_1)$ and $G_2 = (V^*, E_2)$ with labeled vertices are isomorphic to each other iff there is a permutation ψ of the vertices, mapping E_1 to E_2. Thus, the number of non-isomorphic graphs on k vertices and $r/2$ edges is at least:

$$\frac{r^{0.3r}}{k!} \ge \frac{r^{0.3r}}{(r^{0.9})!} > \frac{r^{0.3r}}{r^{0.9r^{0.9}}} \ge r^{r^{0.9}(0.3r^{0.1} - 0.9)} \ge r^{r/10}.$$

□

We are now ready to complete the proof of Observation 1.4. Assume for contradiction that G contains every graph in the family $\mathcal{F}^* = \mathcal{F}_{(20s/\log s)}$ as a minor. Recall that $20s/\log s \ge 2^{20}$. However, from the above two observations, $|\mathcal{F}^*| \ge (20s/\log s)^{20s/(10\log s)}$, while $\mu(G) \le 3^s$. It is immediate to verify that $|\mathcal{F}^*| > \mu(G)$, a contradiction. □

C Proofs Omitted from Section 2

C.1 Proof of Observation 2.1

We assume without loss of generality that $x_1 \ge x_2 \ge \cdots \ge x_r$, and process the integers in this order. When x_i is processed, we add i to A if $\sum_{j\in A} x_j \le \sum_{j \in B} x_j$, and we add it to B otherwise. We claim that at the end of this process, $\sum_{i\in A} x_i, \sum_{i\in B} x_i \ge N/4$ must hold. Indeed, 1 is always added to A. If $x_1 \ge N/4$, then, since $x_1 \le 3N/4$, it is easy to see that both subsets of integers sum up to at least $N/4$. Otherwise, $|\sum_{i\in A} x_i - \sum_{i\in B} x_i| \le \max_i \{x_i\} \le x_1 \le N/4$, and so $\sum_{i\in A} x_i, \sum_{i\in B} x_i \ge N/4$.

C.2 Proof of Claim 2.3.

Our algorithm iteratively removes edges from $T \setminus E'$, until we obtain a connected component of the resulting graph that is an $\alpha/4$-expander. We start with $T' = T \setminus E'$ (notice that T' is not necessarily connected). We also maintain a set E'' of edges that we remove from T', initialized to $E'' = \emptyset$. While T' is not an $\alpha/4$-expander, let (X, Y) be a cut of sparsity less than $\alpha/4$ in T', that is

$|E_{T'}(X,Y)| < \alpha \min(|X|,|Y|)/4$. Assume w.l.o.g. that $|X| \geq |Y|$. Update T' to be $T'[X]$, add the edges of $E(X,Y)$ to E'', and continue to the next iteration.

Assume that the algorithm performs r iterations, and for each $1 \leq i \leq r$, let (X_i, Y_i) be the cut computed by the algorithm in iteration i. Since $|X_i| \geq |Y_i|$, $|Y_i| \leq |V(T')|/2$. At the same time, if we denote $E_i = E'' \cap E(X_i, Y_i)$, then $|E_i| < \alpha|Y_i|/4$. Therefore:

$$|E''| = \sum_{i=1}^{r} |E_i| \leq \alpha \sum_{i=1}^{r} |Y_i|/4.$$

On the other hand, since T is an expander, the total number of edges leaving each set Y_i in T is at least $\alpha|Y_i|$, and all such edges lie in $E' \cup E''$. Therefore:

$$|E'| + |E''| \geq \alpha \sum_{i=1}^{r} |Y_i|/2.$$

Combining both bounds, we get that $|E'| \geq \alpha \sum_{i=1}^{r} |Y_i|/4$. We get that $\sum_{i=1}^{r} |Y_i| \leq \frac{4|E'|}{\alpha}$, and therefore $|V(T')| \geq |V(T)| - \frac{4|E'|}{\alpha}$. □

D Proof of Observation 3.2.

Let τ be any spanning tree of $\hat{G}$, rooted at an arbitrary degree-1 vertex of τ. We start with $\mathcal{U} = \emptyset$. Our algorithm performs a number of iterations, where in each iteration we add one new set $U \subseteq V(\hat{G})$ of vertices to $\mathcal{U}$, such that $\hat{G}[U]$ is connected and $\lfloor |R|/(dr) \rfloor \leq |U \cap R| \leq |R|/r$, and we remove the vertices of U from τ. We execute the iterations as long as $|V(\tau) \cap R| \geq \lfloor |R|/(dr) \rfloor$, after which we terminate the algorithm, and return the current collection $\mathcal{U}$ of vertex subsets.

In order to execute an iteration, we let v be the lowest vertex of τ, such that the subtree τ_v of τ rooted at v contains at least $\lfloor |R|/(dr) \rfloor$ vertices of R. Since the maximum vertex degree in $\hat{G}$ is bounded by d, tree τ_v contains fewer than $d \cdot \lfloor |R|/(dr) \rfloor \leq |R|/r$ vertices of R. We add a new set $U = V(\tau_v)$ of vertices to $\mathcal{U}$, delete the vertices of U from τ, and continue to the next iteration.

Let $\mathcal{U}$ be the final collection of vertex subsets obtained at the end of the algorithm. It is immediate to verify that for every set $U \in \mathcal{U}$, $\hat{G}[U]$ is connected and, from the above discussion, $\lfloor |R|/(dr) \rfloor \leq |U \cap R| \leq |R|/r$. Therefore, $|\mathcal{U}| \geq r$. □

E Proof of Claim 4.6.

Consider the following sequence of vertex subsets. Let $S_0 = Z$, and for all $i > 0$, let S_i contain all vertices of S_{i-1}, and all neighbors of vertices in S_{i-1}. Notice that, if $|S_{i-1}| \leq |V(T)|/2$, then, since T is an α'-expander, there are at least $\alpha'|S_{i-1}|$ edges leaving the set S_{i-1}, and, since the maximum vertex degree in T is at most d, there are at least $\frac{\alpha'|S_{i-1}|}{d}$ vertices that do not belong to S_{i-1}, but are neighbors of vertices in S_{i-1}. Therefore, $|S_i| \geq |S_{i-1}| \left(1 + \frac{\alpha'}{d}\right)$. We claim that there must be an index $i^* \leq \frac{8d}{\alpha'} \log(n/z)$, such that $|S_{i^*}| > |V(T)|/2$. Indeed, otherwise, we get that for $i = \lceil \frac{8d}{\alpha'} \log(n/z) \rceil$:

$$|S_{i^*}| \geq |S_0|\left(1+\frac{\alpha'}{d}\right)^i \geq z \cdot e^{i\alpha'/(2d)} \geq z \cdot e^{4\log(n/z)} > n/2.$$

Here, the second inequality follows from the fact that $(1+1/x)^{2x} > e$ for all $x > 1$. We construct a similar sequence $S'_0, S'_1, \ldots,$ for Z'. Similarly, there is an index $i^{**} \leq \frac{8d}{\alpha'}\log(n/z')$, such that $S'_{i^{**}}$ contains more than half the vertices of T. Therefore, there is a path connecting a vertex of Z to a vertex of Z', whose length is at most $\frac{8d}{\alpha'}(\log(n/z) + \log(n/z'))$. □

F Proof of Lemma 5.1

Recall that we are given a graph $G = (V, E)$, with $|V| \leq n$ and maximum vertex degree at most d, and a parameter $0 < \alpha < 1$. We are also given a collection $\{C_1, \ldots, C_{2r}\}$ of disjoint subsets of V, each containing $q = \lceil cd^2 \log^2 n/\alpha^2 \rceil$ vertices, for some constant c to be fixed later. Our goal is to either find a set $\mathcal{Q} = \{Q_1, \ldots, Q_r\}$ of disjoint paths, such that for each $1 \leq j \leq r$, path Q_j connects C_j to C_{j+r}; or compute a cut (S, S') in G of sparsity less than α.

We use a standard definition of multicommodity flow. A *flow* f consists of a collection $\mathcal{P}$ of paths in G, called *flow-paths*, and, for each path $P \in \mathcal{P}$, an associated flow value $f(P) > 0$. The *edge-congestion* of f is the maximum amount of flow passing through any edge, that is, $\max_{e\in E}\left\{\sum_{\substack{P\in\mathcal{P}:\\ e\in P}} f(P)\right\}$. We say that the flow in f causes *no edge-congestion* iff the edge-congestion due to f is at most 1. Similarly, the *vertex congestion* of f is the maximum flow passing through any vertex, that is, $\max_{v\in V}\left\{\sum_{\substack{P\in\mathcal{P}:\\ v\in P}} f(P)\right\}$. If a path P does not lie in $\mathcal{P}$, then we implicitly set $f(P) = 0$. For any pair $s, t \in V$ of vertices, let $\mathcal{P}(s,t)$ be the set of all paths connecting s to t in G. We say that f *transfers* z *flow units between* s *and* t iff $\sum_{P\in\mathcal{P}(s,t)} f(P) \geq z$.

The following theorem is a consequence of Theorem 18 from [LR99] that we prove after completing the proof of Lemma 5.1.

Theorem F.1 *There is an efficient randomized algorithm, that, given a graph $G = (V, E)$ with $|V| = n$ and maximum vertex degree at most d, and a parameter $0 < \alpha < 1$, together with a (possibly partial) matching $\mathcal{M}$ over the vertices of G, computes one of the following:*

- *either a collection $\mathcal{Q}' = \{Q(u,v) \mid (u,v) \in \mathcal{M}\}$ of paths, such that for all $(u,v) \in \mathcal{M}$, path $Q(u,v)$ connects u to v; the paths in $\mathcal{Q}'$ with high probability cause vertex-congestion at most $\eta = O(d\log n/\alpha)$, and the length of every path in $\mathcal{Q}$ is at most $L = O(d\log n/\alpha)$; or*
- *a cut (S, S') in G of sparsity less than α.*

We are now ready to complete the proof of Lemma 5.1. We construct a matching $\mathcal{M}$ over the vertices of V, as follows. For each $1 \leq j \leq r$, we add an arbitrary matching $\mathcal{M}_j$, containing q edges, between the vertices of C_j and the vertices C_{j+r}. We then set $\mathcal{M} = \bigcup_{j=1}^r \mathcal{M}_j$. We apply the algorithm from Theorem F.1 to the graph G, parameter α and the matching $\mathcal{M}$. If the algorithm returns a cut of sparsity less than α, we terminate the algorithm and return the cut. Therefore, we assume from now on that the algorithm returns a set $\mathcal{Q}'$ of paths with the following properties:

- For each $j \in [r]$, there is a subset $\mathcal{Q}'_j \subseteq \mathcal{Q}'$ of q paths connecting vertices of C_j to vertices of C_{j+r};

- All paths in $\mathcal{Q}'$ have length at most $L = O(d \log n/\alpha)$; and
- With high probability, every vertex of G participates in at most $\eta = O(d \log n/\alpha)$ paths of $\mathcal{Q}'$.

If the vertex-congestion caused by the paths in $\mathcal{Q}'$ is greater than η, the algorithm terminates with a failure. Therefore, we assume from now on that the paths in $\mathcal{Q}'$ cause vertex-congestion at most η. We use the constructive version of the Lovasz Local Lemma by Moser and Tardos [MT10] in order to select one path from each set $\mathcal{Q}'_j$, so that the resulting paths are node-disjoint with high probability. The next theorem summarizes the symmetric version of the result of [MT10].

Theorem F.2 ([MT10]) *Let X be a finite set of mutually independent random variables in some probability space. Let $\mathcal{A}$ be a finite set of bad events determined by these variables. For each event $A \in \mathcal{A}$, let $\mathrm{vbl}(A) \subseteq X$ be the unique minimal subset of variables determining A, and let $\Gamma(A) \subseteq \mathcal{A}$ be a subset of bad events B, such that $A \neq B$, but $\mathrm{vbl}(A) \cap \mathrm{vbl}(B) \neq \emptyset$. Assume further that for each $A \in \mathcal{A}$, $|\Gamma(A)| \leq D$, $\mathbf{Pr}[A] \leq p$, and $ep(D+1) \leq 1$. Then there is an efficient randomized algorithm that computes an assignment to the variables of X, such that with high probability none of the events in $\mathcal{A}$ holds.*

For each $1 \leq i \leq r$, we choose one of its paths $Q_i \in \mathcal{Q}_i$ independently at random. We let z_i be the random variable indicating which path has been chosen. For every pair $Q, Q' \in \mathcal{Q}'$ of intersecting paths, such that Q, Q' belong to distinct sets $\mathcal{Q}'_i, \mathcal{Q}'_j$ let $\mathcal{E}(Q, Q')$ be the bad event that both these paths were selected. Notice that the probability of $\mathcal{E}(Q, Q')$ is $1/q^2$. Notice also that $\mathrm{vbl}(\mathcal{E}(Q, Q')) = \{z_i, z_j\}$, where $Q \in \mathcal{Q}'_i, Q' \in \mathcal{Q}'_j$. There are at most $qL\eta$ events $\mathcal{E}(\hat{Q}, \hat{Q}')$, with $z_i \in \mathrm{vbl}(\mathcal{E}(Q, Q'))$: set $\mathcal{Q}'_i$ contains q paths; each of these paths has length at most L, so there are at most qL vertices that participate in the paths in $\mathcal{Q}'_i$. Each such vertex may be shared by at most η other paths. Similarly, there are at most $qL\eta$ events $\mathcal{E}(\hat{Q}, \hat{Q}')$, with $z_j \in \mathrm{vbl}(\mathcal{E}(Q, Q'))$. Therefore, $|\Gamma(\mathcal{E}(Q, Q'))| \leq 2qL\eta$. Let $D = 2qL\eta$. It now only remains to show that $(D+1)ep \leq 1$. Indeed,

$$(D+1)ep = \frac{O(qL\eta)}{q^2} = \frac{O(L\eta)}{q} = O\left(\frac{d^2 \log^2 n}{\alpha^2 q}\right).$$

By choosing the constant c in the definition of q to be large enough, we can ensure that $(D+1)ep \leq 1$ holds. Using the algorithm from Theorem F.2, we obtain a collection $\mathcal{Q} = \{Q_1, \ldots, Q_r\}$ of paths in G, where for each $j \in [r]$, path Q_j connects a vertex of C_j to a vertex of C_{j+r}, and with high probability the resulting paths are disjoint. This completes the proof of Lemma 5.1, except for the proof of Theorem F.1 that we provide next.

F.1 Proof of Theorem F.1.

We use a slight adaptation of Theorem 18 from [LR99].

Theorem F.3 (Adaptation of Theorem 18 from [LR99]) *There is an efficient algorithm, that, given a n-vertex graph G with maximum vertex degree at most d, together with a parameter $0 < \alpha < 1$ computes one of the following:*

- *either a flow f in G, with every pair of vertices in G transferring $\frac{\alpha}{64 n \log n}$ flow units to each other with no edge-congestion, such that every flow-path has length at most $\frac{64 d \log n}{\alpha}$; or*
- *a cut (S, S') in G of sparsity less than α.*

We provide the proof of Theorem F.3 below, after completing the proof of Theorem F.1 using it.

We apply Theorem F.3 to the graph G and the parameter α. If the algorithm returns a cut (S, S') of sparsity less than α, then we terminate the algorithm and return this cut. Therefore, we assume from now on that the algorithm returns the flow f. Let f' be a flow obtained from f by scaling it up by factor $64 \log n/\alpha$, so that every pair of vertices in G now sends $1/n$ flow units to each other, with total edge-congestion at most $64 \log n/\alpha$.

We start by showing that there is a multi-commodity flow f^*, where every pair $(u, v) \in \mathcal{M}$ of vertices sends one flow unit to each other simultaneously, on flow-paths of length at most $128d \log n/\alpha$, with total vertex-congestion at most $128d \log n/\alpha$. Let $(u, v) \in \mathcal{M}$ be any pair of vertices. The new flow between u and v is defined as follows: u sends $1/n$ flow units to every vertex of G, using the flow f', and v collects $1/n$ flow units from every vertex of G, using the flow f'. In other words, the flow f^* between u and v is obtained by concatenating all flow-paths in f' originating at u with all flow-paths in f' terminating at v. It is easy to see then that every flow-path in f' is used at most twice: once by each of its endpoints; all flow-paths in f^* have length at most $128d \log n/\alpha$; and the total edge-congestion due to flow f^* is at most $128 \log n/\alpha$. Since the maximum vertex degree in G is at most d, flow f^* causes vertex-congestion at most $128d \log n/\alpha$.

Next, for every pair $(u, v) \in \mathcal{M}$, we select one path $Q(u, v) \in \mathcal{P}(u, v)$ at random, where a path $P \in \mathcal{P}(u, v)$ is selected with probability $f^*(P)$ – the amount of flow sent on P by f^*. We then let $\mathcal{Q}' = \{Q(u, v) \mid (u, v) \in \mathcal{M}\}$. Notice that the length of every path in $\mathcal{Q}'$ is at most $128d \log n/\alpha$. It remains to show that the total vertex-congestion due to paths in $\mathcal{Q}'$ is at most $O(d \log n/\alpha)$ with high probability. This is done by standard techniques. Consider some vertex $x \in V$. We say that the bad event $\mathcal{E}(x)$ happens if more than $8 \cdot 128d \log n/\alpha$ paths of $\mathcal{Q}$ use the vertex x. We use the following variation of the Chernoff bound (see [DP09]):

Theorem F.4 *Let $X_1, \ldots, X_n$ be independent random variables taking values in $[0, 1]$, let $X = \sum_i X_i$, and let $\mu = \mathbf{E}[X]$. Then for all $t > 2e\mu$, $\mathbf{Pr}[X > t] \le 2^{-t}$.*

It is easy to see that the expected number of paths in $\mathcal{Q}'$ that contain x is at most $128d \log n/\alpha$, and so the probability of $\mathcal{E}(x)$ is bounded by $1/n^4$. From the Union Bound, the probability that any such event happens for any vertex $x \in V$ is bounded by $1/n^3$. Therefore, with high probability, every vertex of G belongs to $2^{10} d \log n/\alpha = O(d \log n/\alpha)$ paths in $\mathcal{Q}'$. This finishes the proof of Theorem F.1 except for the proof of Theorem F.3, that we prove in the next sub-section. □

F.2 Proof of Theorem F.3

The proof follows closely that of [LR99]; we provide it here for completeness. Recall that we are given a graph $G = (V, E)$ with maximum vertex-degree at most d, $|V| = n$ and a parameter $0 < \alpha < 1$. We let $L = 64d \log n/\alpha$. For every pair u, v of vertices in V, let $\mathcal{P}^{\le L}(u, v)$ be the set of all paths in G between u and v that contain at most L vertices. We employ standard linear program for uniform multicommodity flow:

$$
\begin{array}{lll}
\text{(LP-1)} & \max & f^* \\
\text{s.t.} & & \\
& \sum_{P\in\mathcal{P}^{\leq L}(u,v)} f(P) \geq f^* & \forall u,v\in V \\
& \sum_{u,v\in V}\sum_{\substack{P\in\mathcal{P}^{\leq L}(u,v):\\ e\in P}} f(P)\leq 1 & \forall e\in E \\
& f(P)\geq 0 & \forall u,v\in V; \forall P\in\mathcal{P}^{\leq L}(u,v)
\end{array}
$$

In general, the dual of the standard relaxation of the uniform multicommodity flow problem is the problem of assigning lengths $\ell(e)$ to the edges $e \in E$, so as to minimize $\sum_e \ell(e)$, subject to the constraint that the total sum of all pairwise distances between pairs of vertices is at least 1, where the distance between pairs of vertices is defined with respect to ℓ.

In our setting, given lengths $\ell(e)$ on edges $e \in E$, we need to use L-hop bounded distances between vertices, defined as follows: for all $u, v \in V$, if $\mathcal{P}^{\leq L}(u,v) \neq \emptyset$, then we let:

$$
D_\ell^{\leq L}(u,v) = \min_{P\in\mathcal{P}^{\leq L}(u,v)} \left\{\sum_{e\in P} \ell(e)\right\};
$$

otherwise, we set $D_\ell^{\leq L}(u,v) = \infty$. The dual of (LP-1) can now be written as follows:

$$
\begin{array}{lll}
\text{(LP-2)} & \min & \sum_{e\in E} \ell(e) \\
\text{s.t.} & & \\
& \sum_{u,v\in V} D_\ell^{\leq L}(u,v) \geq 1 & \\
& \ell(e)\geq 0 & \forall e\in E
\end{array}
$$

Even though Linear Programs (LP-1) and (LP-2) are of exponential size, they can be solved efficiently using standard techniques (that is, edge-based flow formulation). Let f^*_{OPT} be the value of the optimal solution to (LP-1). We let $W^* = \frac{d}{nL} = \frac{\alpha}{64n\log n}$. If $f^*_{\mathsf{OPT}} \geq W^*$, then we return the flow f corresponding to the optimal solution of (LP-1); it is immediate to verify that it satisfies all requirements. Therefore, we assume from now on that $f^*_{\mathsf{OPT}} < W^*$. We will provide an efficient algorithm to compute a cut (S, S') in G of sparsity less than α.

Given a length function $\ell : E \mapsto \mathbb{R}_{\geq 0}$, we denote by $W(\ell) = \sum_{e\in E} \ell(e)$ the total 'weight' of ℓ. We need the following definition.

Definition 8 *Given an integer r and a length function $\ell(e)$ on edges $e \in E$, the r-hop bounded diameter of G is* $\max_{u,v\in V}\left\{D_\ell^{\leq r}(u,v)\right\}$.

Consider the optimal solution $\ell_{\mathsf{OPT}} : E \to \mathbb{R}^+$ to (LP-2). Observe that, by the strong duality, the value of the solution $W(\ell_{\mathsf{OPT}}) = f^*_{\mathsf{OPT}}$, and so $W(\ell_{\mathsf{OPT}}) < W^*$ holds.

We define a new solution ℓ to (LP-2) as follows: for each edge e, we let $\ell(e) = \ell_{\mathsf{OPT}}(e) \cdot \frac{W^*}{W(\ell_{\mathsf{OPT}})}$. Since $W^* > W(\ell_{\mathsf{OPT}})$, it immediate to verify that we obtain a valid solution to (LP-2), of value $W(\ell) = W^*$.

Moreover, the constraint governing the sum of pairwise L-hop bounded distances is now satisfied with strict inequality:

$$\sum_{u,v} D_\ell^{\leq L}(u,v) > 1. \tag{2}$$

The lengths $\ell(e)$ on edges are fixed from now on, and we denote $D_\ell^{\leq L}$ by $D^{\leq L}$ from now on. We will also use the distance function $D_\ell^{\leq L/4}$, that we denote by $D^{\leq L/4}$ from now on.

We use the following lemma.

Lemma F.5 (Adaptation of Corollary 20 from [LR99]) *There is an efficient algorithm, that, given a graph $G=(V,E)$, a parameter $0<\alpha<1$ and any edge length function $\ell : E \mapsto \mathbb{R}_{\geq 0}$ of total weight $W(\ell)=\sum_{e\in E}\ell(e) \leq \frac{\alpha}{64n\log n}$, returns one of the following:*

- *either a subset $T \subseteq V$ of at least $\left\lceil \frac{2|V|}{3} \right\rceil$ vertices, such that, for $r=\frac{|E|}{2n^2W(\ell)}$, the r-hop bounded diameter of $G[T]$ is at most $\frac{1}{2n^2}$; or*
- *a cut (S,S') in G of sparsity less than α.*

We complete the proof of Lemma F.5 later, after we complete the proof of Theorem F.3 using it. Recall that $W(\ell)=W^*=\frac{\alpha}{64n\log n}$. We apply the algorithm from Lemma F.5 to graph G, with parameter α and distance function ℓ.

If the algorithm returns a cut (S,S') of sparsity less than α, we terminate the algorithm and return this cut. Therefore, we assume from now on that the algorithm from Lemma F.5 returns a subset $T\subseteq V$ of at least $2|V|/3$ vertices such that $G[T]$ has r-hop bounded diameter at most $\frac{1}{2n^2}$, where $r=\frac{|E|}{2n^2W(\ell)}$. Observe that for all $r'>r$, for every pair u,v of vertices, $D^{\leq r'}(u,v)\leq D^{\leq r}(u,v)$. Observe also that:

$$r=\frac{|E|}{2n^2W(\ell)} \leq \frac{\frac{dn}{2}}{2n^2\frac{d}{nL}} = \frac{L}{4}.$$

Therefore, the $L/4$-hop bounded diameter of $G[T]$ is at most $\frac{1}{2n^2}$.

For convenience, for a subset $S\subseteq V$ of vertices and a vertex $u\in V$, we denote by $D^{\leq L/4}(u,S) := \min_{v\in S} D^{\leq L/4}(u,v)$. We use the following lemma.

Lemma F.6 (Adaptation of Lemma 21 from [LR99]) *There is an efficient algorithm, that, given a graph $G=(V,E)$, a parameter $0<\alpha<1$, any edge length function $\ell : E\mapsto \mathbb{R}_{\geq 0}$, a length parameter $L\geq \frac{2d\ln n}{\alpha}$ and a subset $T\subseteq V$ of at least $\lceil 2|V|/3\rceil$ vertices, such that $\sum_{v\in V} D^{\leq L}(v,T) > \frac{4W(\ell)}{\alpha}$, returns a cut (S,S') of V with sparsity less than α.*

We prove Lemma F.6 later, after we complete the proof of Theorem F.3 using it.

First, we claim that $\sum_{v\in V} D^{\leq L/4}(v,T) > \frac{4W^*}{\alpha}$. Indeed, assume for contradiction otherwise, that is:

$$\sum_{v\in V} D^{\leq L/4}(v,T) \leq \frac{4W^*}{\alpha} = \frac{4}{\alpha}\cdot\frac{\alpha}{64n\log n} = \frac{1}{16n\log n}.$$

Recall that the $L/4$-hop bounded diameter of $G[T]$ is at most $\frac{1}{2n^2}$. From the triangle inequality, for any pair $u,v\in V$ of vertices:

$$D^{\leq L}(u,v) \leq D^{\leq L/4}(u,T) + D^{\leq L/4}(v,T) + \frac{1}{2n^2}.$$

Hence,

$$\begin{aligned}\sum_{u,v\in V} D^{\leq L}(u,v) &\leq \sum_{u,v\in V}\left(D^{\leq L/4}(u,T) + D^{\leq L/4}(v,T) + \frac{1}{2n^2}\right)\\ &\leq \frac{1}{2} + 2n\sum_{u\in V} D^{\leq L/4}(u,T)\\ &\leq \frac{1}{2} + 2n\frac{1}{16n\log n} = \frac{1}{2} + \frac{1}{8\log n} < 1,\end{aligned}$$

contradicting the fact that ℓ is a valid solution to (LP-2). Therefore, $\sum_{u\in V} D^{L/4}(v,T) > \frac{4W^*}{\alpha}$ must hold. Moreover, notice that $\frac{L}{4} = \frac{16d\log n}{\alpha} \geq \frac{2d\ln n}{\alpha}$ holds. We now apply the algorithm from Lemma F.6 to G, with parameters α and $L/4$, edge length function ℓ and the subset T of vertices, to obtain a cut (S,S') of V with sparsity less than α. This completes the proof of Theorem F.3, except for the proofs of Lemma F.5 and Lemma F.6 that we provide in the next subsection.

F.3 Proof of Lemma F.5

We start with the following definition:

Definition 9 *Given a graph $G=(V,E)$, a partition of G into components is a collection $\mathcal{G} = \{G[V_1],\ldots,G[V_z]\}$ of vertex-induced subgraphs such that $\bigcup_{i\in[z]} V_i = V$ and for every $i\neq j$, $V_i \cap V_j = \emptyset$.*

We use the following lemma, that we prove later for completeness after completing the proof of Lemma F.5 using it.

Lemma F.7 (Adaptation of Lemma 19 from [LR99]) *There is an efficient algorithm, that, given a graph $G=(V,E)$, a parameter $\Delta > 0$, and any edge length function $\ell : E \mapsto \mathbb{R}_{\geq 0}$, partitions G into components $\mathcal{G} = \{G[V_1],\ldots,G[V_z]\}$ such that the following holds:*

- *For each $G[V_i]\in\mathcal{G}$, the r'-hop bounded diameter of $G[V_i]$ is at most Δ, for $r' = \Delta|E|/W(\ell)$; and*
- $\sum_{i<j} |E(V_i,V_j)| < 8W(\ell)\log n/\Delta$.

We use Lemma F.7 with $\Delta = \frac{1}{2n^2}$ and edge length function ℓ to obtain a collection $\mathcal{G} = \{G[V_1],\ldots,G[V_z]\}$ of components. Notice that $r' = \frac{\Delta|E|}{W(\ell)} = \frac{|E|}{2n^2 W(\ell)} = r$, so the r-hop bounded diameter of each subgrpaph $G[V_i]$ is at most $1/n^2$.

If, for some subgraph $G[V_{i^*}]\in\mathcal{G}$, $|V_{i^*}| \geq \frac{2|V|}{3}$, then we return V_{i^*}. Otherwise, we use Observation 2.1, to obtain a partition of the graphs in $\mathcal{G}$ into two subsets, $\mathcal{G}'$ and $\mathcal{G}''$, such that, if we let $S = \bigcup_{G_i\in\mathcal{G}'} V(G_i)$, and $S' = \bigcup_{G_i\in\mathcal{G}'} V(G_i)$, then $|S|,|S'| \geq |V|/4$ and $|E(S,S')| < \frac{8W(\ell)\log n}{\Delta} = 16W(\ell)n^2\log n$. Therefore, the sparsity of the cut (S,S') is less than:

$$\frac{16W(\ell)n^2\log n}{n/4} = 64W(\ell)n\log n \leq 64n\log n\cdot\frac{\alpha}{64n\log n} = \alpha.$$

This completes the proof of Lemma F.5 except for the proof of Lemma F.7 that we provide next.

Proof of Lemma F.7. If $\Delta < \frac{8W(\ell)\log n}{|E|}$, we output $\mathcal{G} = \{G[\{v\}] \mid v \in V\}$. Notice that for each $G[V_i]$, we have $G[V_i] = G[\{v_i\}]$ for some $v_i \in V$. Hence, the r'-hop bounded diameter of $G[V_i]$ is 0, and we have $\sum_{i<j} |E(V_i, V_j)| = |E| < \frac{8W(\ell)\log n}{\Delta}$ as required. Therefore, we assume from now on that $\Delta \geq \frac{8W(\ell)\log n}{|E|} > \frac{8W(\ell)\ln n}{|E|}$ holds. For convenience, we denote $\epsilon := \frac{2W(\ell)\ln n}{\Delta|E|}$. Notice that $\epsilon \leq 1/4$ holds.

Consider an auxiliary graph $G^+ = (V^+, E^+)$ obtained from G by replacing each edge e with a path consisting of $\lceil |E|\ell(e)/W(\ell) \rceil$ edges. Notice that $|E^+| \leq 2|E|$. For simplicity, we identify the common vertices of G and G^+. The following observation is now immediate:

Observation F.8 *For any path of length γ in G^+, the corresponding path in G has length at most $\frac{W(\ell)\gamma}{|E|}$.*

Next, we iteratively partition vertices of G^+ into $V_0^+, V_1^+, \ldots$, and the required partition of G into components will be given by $G[V_1] = G[V_1^+ \cap V], G[V_2] = G[V_2^+ \cap V], \ldots,$. We start with $V_0^+ = \emptyset$ and then iterate. We now show how to compute V_{i+1}^+ given $V_0^+, \ldots, V_i^+$.

We denote $V_i^* := V^+ \setminus \bigcup_{j \leq i} V_j^+$. If $V \cap V_i^* = \emptyset$, we have computed the desired partition and the algorithm terminates. Thus, we assume from now on that there is a vertex $v_{i+1} \in V_i^*$. For every integer $j \geq 0$, we denote by B_j^{i+1} the subset of vertices $u \in V_i^*$, such that there is some path of length at most j connecting v_{i+1} and u in $G^+[V_i^*]$.

We let $C_j := \frac{2|E|}{n} + |E_G[B_j^{i+1}]|$ for every integer $j \geq 0$. Let j_{i+1}^* be the smallest $j \geq 0$ such that $C_{j+1} < (1+\epsilon)C_j$. Notice that some such j_{i+1}^* must exist, since $\epsilon > 0$ and $C_{j+1} = C_j$ for $j \to \infty$. We set $V_{i+1}^+ = B_{j_{i+1}^*}^{i+1}$ and proceed to the next iteration. The following observation is now immediate:

Observation F.9 *For every index $i > 0$, $V_i^+ \cap V \neq \emptyset$ and $|E(V_i^+, V_i^*)| < \epsilon \left(\frac{2|E|}{n} + \left|E[V_i^+]\right|\right)$.*

Proof: Notice that for every index $i > 0$ and $j \geq 0$, we have $v_i \in B_j^i$. Thus, $v_i \in V_i^+ \cap V$, and hence $V_i^+ \cap V \neq \emptyset$. From our construction, we have

$$\frac{2|E|}{n} + \left|E[B_{j_i^*+1}^i]\right| < (1+\epsilon)\left(\frac{2|E|}{n} + \left|E[B_{j_i^*}^i]\right|\right).$$

Equivalently:

$$\left|E[B_{j_i^*+1}^i]\right| - \left|E[B_{j_i^*}^i]\right| < \epsilon\left(\frac{2|E|}{n} + \left|E[B_{j_i^*}^i]\right|\right).$$

Therefore,

$$|E(V_i^+, V_i^*)| \leq \left|E[B_{j_i^*+1}^i]\right| - \left|E[B_{j_i^*}^i]\right| < \epsilon\left(\frac{2|E|}{n} + \left|E[B_{j_i^*}^i]\right|\right) = \epsilon\left(\frac{2|E|}{n} + \left|E[V_i^+]\right|\right).$$

□

The following two claims will complete the proof of Lemma F.7.

Claim F.10 $\sum_{i<j} |E(V_i, V_j)| < \frac{8W(\ell)\log n}{\Delta}$.

Proof:

$$\sum_{i<j} |E(V_i, V_j)| = \sum_{i>0} \left|E\left(V_i, \bigcup_{j>i} V_j\right)\right| \leq \sum_{i>0} |E(V_i^+, V_i^*)| < \sum_{i>0} \epsilon\left(\frac{2|E|}{n} + |E(|V_i^+|)|\right)$$

$$\leq \epsilon\left(2|E| + |E^+|\right) \leq 4|E|\epsilon = \frac{8W(\ell)\ln n}{\Delta} < \frac{8W(\ell)\log n}{\Delta}.$$

Here, the second inequality follows from Observation F.9 and the penultimate inequality follows from the fact that $|E^+| \leq 2|E|$. □

Claim F.11 *For each $G[V_i]$, the r'-hop bounded diameter of $G[V_i]$ is at most Δ, for $r' = \frac{\Delta|E|}{W(\ell)}$.*

Proof: We claim that it suffices to show that, for each $G[V_i]$, the diameter of $G^+[V_i^+]$ is at most $r' = \frac{\Delta|E|}{W(\ell)}$. Indeed, if this is the case, Observation F.8 implies that the r'-hop bounded diameter of $G[V_i]$ is at most $\frac{W(\ell)r'}{|E|} = \Delta$. Notice that, in order to show that the diameter of $G^+[V_i^+]$ is at most r', it suffices to show that $j_i^* \leq \frac{r'}{2} = \frac{\Delta|E|}{2W(\ell)}$. Fix any index i and the corresponding graph $G^+[V_i^+]$. If $j_i^* \neq 0$, we must have:

$$2|E| \geq |E^+| \geq |E(V_i^+)| > (1+\epsilon)^{j_i^*}\frac{2|E|}{n}.$$

Therefore, $(1+\epsilon)^{j_i^*} < n$ must hold, and so:

$$j_i^* < \frac{\ln n}{\epsilon} = \frac{\Delta|E|}{2W(\ell)}.$$

(We have used the fact that $\epsilon < 1/4$). □ □

F.4 Proof of Lemma F.6

Similarly to the proof of Lemma F.6, consider an auxiliary graph $G^+ = (V^+, E^+)$ obtained from G by replacing each edge e with a path consisting of $\lceil |E|\ell(e)/W(\ell)\rceil$ edges. Notice that $|E^+| \leq 2|E|$. For simplicity, we identify the common vertices of G and G^+. Given a subset $S \subseteq V(G^+)$ of vertices, we denote by $N(S)$ the set of all vertices $v \in V(G^+)$ such that $v \notin S$, but v has a neighbor in S.

Next, we iteratively partition the vertices of G^+ into layers, $V_0^+, V_1^+, \ldots$, and for each $i \geq 0$, we define the corresponding graph $G_i^+ = G^+[V_i^+]$, as follows. We start with $V_0^+ = T$, $G_0^+ = G^+[T]$ and then iterate. We now show how to compute V_{i+1}^+ and G_{i+1}^+, given V_i^+ and G_i^+, assuming that $V_i^+ \neq V^+$ (otherwise, the algorithm terminates).

Let $E_i := \delta_{G^+}(V_i^+)$ and $C_i := |E_i|$. We partition E_i into two subsets: set E_i' containing all edges (u,v) with $u \in V_i^+$, such that v is a vertex of the original graph G; and set E_i'' containing all remaining edges. Let $C_i' = |E_i'|$, and let $C_i'' = |E_i''|$. We distinguish between the following two cases:

- **Case 1:** $C_i' \geq C_i/2$. In this case, we let V_{i+1}^+ contain all vertices of $V_i^+ \cup N(V_i^+)$. We also set $G_{i+1}^+ = G^+[V_{i+1}^+]$. Notice that in this case, $|E[G_{i+1}^+] \setminus E[G_i^+]| \geq C_i$.
- **Case 2:** $C_i'' > C_i/2$. In this case, we let V_{i+1}^+ only contain the vertices of V_i^+, and those vertices of $N(V_i^+)$ that do not lie in the original graph G, that is:

$$V_{i+1}^+ = V_i^+ \cup (N(V_i^+) \setminus V(G)).$$

  As before, we set $G_{i+1}^+ = G^+[V_{i+1}^+]$. Notice that in this case, $E[G_{i+1}^+] \setminus E[G_i^+]$ contains all edges of E_i'', and so $|E[G_{i+1}^+] \setminus E[G_i^+]| \geq C_i'' > C_i/2$.

From the above discussion we obtain the following observation:

Observation F.12 *For each level i, $|E(G^+_{i+1}) \setminus E(G^+_i)| \geq \frac{C_i}{2}$, and in particular $\sum_i C_i \leq 2|E^+|$.*

For each level i, let $n_i = |V(G) \backslash V_i^+|$ – the number of vertices of the original graph G that do not lie in V_i^+. Recall that $|T| \geq \lceil 2|V|/3 \rceil$, and so for all i, $n_i \leq |V|/3 \leq |V|/2$. Moreover, $C_i = |\delta_{G^+}(V_i^+)| \geq |\delta_G(V \cap V_i^+)|$.

If, for any level i, $C_i < \alpha n_i$, then we return the cut $(V \cap V_i^+, V \backslash V_i^+)$; it is immediate to see that its sparsity is less than α. Therefore, we assume from now on, that for all i, $C_i \geq \alpha n_i$. We will reach a contradiction by showing that $\sum_{v \in V} D^{\leq L}(v, T) \leq \frac{4W(\ell)}{\alpha}$ must hold. In order to do so, we use the following two claims.

Claim F.13 *The number of indices i for which Case 1 is invoked is at most L.*

Proof: Let i be an index for which Case 1 is invoked, so $C'_i \geq C_i/2$. Recall that we have assumed that $C_i \geq \alpha n_i$. Since the maximum vertex-degree of G is bounded by d, the number of new vertices of V that are added to V^+_{i+1} is at least $\frac{C'_i}{d} \geq \frac{\alpha n_i}{2d}$. Therefore, $n_{i+1} \leq n_i(1 - \frac{\alpha}{2d})$, and the total number of indices i in which Case 1 is invoked must be bounded by $\frac{2d \ln n}{\alpha} \leq L$. □

Claim F.14 $\sum_i n_i \leq \frac{4|E|}{\alpha}$.

Proof: Recall that we have assumed $C_i \geq \alpha n_i$ for all i. Thus,

$$\sum_i n_i \leq \sum_i \frac{C_i}{\alpha} = \frac{\sum_i C_i}{\alpha} \leq \frac{2|E^+|}{\alpha} \leq \frac{4|E|}{\alpha}.$$

Here, the second-last inequality follows from Observation F.12 and the last inequality follows from the fact that $|E^+| \leq 2|E|$. □

For each vertex $v \in V \backslash T$, let i_v be the unique index, such that $v \in V(G^+_i)$ and $v \notin V(G^+_{i-1})$. For the remaining vertices $v \in T$, we set $i_v = 0$. Notice that v must be connected by an edge to a vertex u with $i_u < i_v$. Therefore, we can construct a path $P^+_v = (v_0, v_1, \ldots, v_r)$ in G^+, where $v_0 \in T$, $v_r = v$, and for all $1 \leq j \leq r$, $i_{v_{j-1}} < i_{v_j}$.

Let P_v be the path corresponding to P^+_v in the original graph G. Since we invoke Case 1 at most L times, it is easy to verify that P_v contains at most L edges. Moreover:

$$D^{\leq L}(v, T) \leq \sum_{e \in P_v} \ell(e) \leq \sum_{e \in P_v} \frac{W(\ell)}{|E|} \left\lceil \frac{|E|\ell(e)}{W(\ell)} \right\rceil = \frac{W(\ell)}{|E|} |E(P^+_v)| \leq i_v \frac{W(\ell)}{|E|}.$$

Altogether:

$$\sum_v D^{\leq L}(v, T) \leq \frac{W(\ell)}{|E|} \sum_v i_v = \frac{W(\ell)}{|E|} \sum_i n_i \leq \frac{4W(\ell)}{\alpha},$$

where the last inequality follows from Claim F.14. This contradicts the assumption that $\sum_v D^{\leq L}(v, T) > \frac{4W(\ell)}{\alpha}$, completing the proof of Lemma F.6.

G Proofs Omitted from Section 6

G.1 Proof of Claim 6.6

The proof is very similar to the proof of Claim 2.3. The algorithm iteratively removes edges from $G \setminus E'$, until we obtain a connected component of the resulting graph that is an $\Omega\left(\frac{\alpha^2}{d}\right)$-expander. We start with $G' = G \setminus E'$ (notice that G' is not necessarily connected). We also maintain a set E'' of edges that we remove from G', initialized to $E'' = \emptyset$. We then perform a number of iterations. In every iteration, we apply Theorem 2.2 to G', and obtain a cut (X, Y) in G'. If $|E_{G'}(X, Y)| \geq \alpha \cdot \min(|X|, |Y|)/4$, then, from Theorem 2.2, G' is an $\Omega\left(\frac{\alpha^2}{d}\right)$-expander. We terminate the algorithm and return G'. We later show that $|V(G')| \geq |V| - \frac{4|E'|}{\alpha}$. Assume now that $|E_{G'}(X, Y)| < \alpha \cdot \min(|X|, |Y|)/4$, and assume w.l.o.g. that $|X| \geq |Y|$. Update G' to be $G'[X]$, add the edges of $E(X, Y)$ to E'', and continue to the next iteration. Clearly, at the end of the algorithm, we obtain a graph G' that is an $\Omega\left(\frac{\alpha^2}{d}\right)$-expander. It only remains to show that $|V(G')| \geq |V| - \frac{4|E'|}{\alpha}$. The remainder of the analysis is identical to the analysis of Claim 2.3.

Assume that the algorithm performs r iterations, and for each $1 \leq i \leq r$, let (X_i, Y_i) be the cut computed by the algorithm in iteration i. Since $|X_i| \geq |Y_i|$, $|Y_i| \leq |V(G)|/2$. At the same time, if we denote $E_i = E'' \cap E(X_i, Y_i)$, then $|E_i| < \alpha |Y_i|/4$. Therefore:

$$|E''| = \sum_{i=1}^{r} |E_i| < \alpha \sum_{i=1}^{r} |Y_i|/4.$$

On the other hand, since G is an α-expander, the total number of edges leaving each set Y_i in G is at least $\alpha|Y_i|$, and all such edges lie in $E' \cup E''$. Therefore:

$$|E'| + |E''| \geq \alpha \sum_{i=1}^{r} |Y_i|/2.$$

Combining both bounds, we get that $|E'| \geq \alpha \sum_{i=1}^{r} |Y_i|/4$, and so $\sum_{i=1}^{r} |Y_i| \leq \frac{4|E'|}{\alpha}$. Therefore, $|V(G')| = |V| - \sum_{i=1}^{r} |Y_i| \geq |V| - \frac{4|E'|}{\alpha}$.

G.2 Proof of Claim 6.9

We can compute the largest-cardinality set of disjoint paths connecting vertices of A to vertices of B in G using standard maximum s–t flow computation and the integrality of flow. Therefore, it is sufficient to show that there exists a set of $\lceil \alpha z/d \rceil$ disjoint paths connecting A to B in G.

Assume otherwise. Then, from Menger's theorem, there is a set Z of fewer than $\alpha z/d$ vertices in G, such that $G \setminus Z$ contains no path from a vertex of $A \setminus Z$ to a vertex of $B \setminus Z$. Let E' be the set of all edges of G incident to the vertices of Z. Since the maximum vertex degree in G is at most d, $|E'| < \alpha z$. Therefore, graph $G \setminus E'$ contains no path connecting a vertex of A to a vertex of B. Let X be the union of all connected components of $G \setminus E'$ containing the vertices of A, and let $Y = V(G) \setminus X$. Then $|E(X, Y)| \leq |E'| < \alpha z \leq \alpha \cdot \min\{|X|, |Y|\}$, contradicting the fact that G is an α-expander.

G.3 Proof of Theorem 6.10

The main tool that we use for the proof of Theorem 6.10 is the following theorem, whose proof appeared in [CC16]; we include the proof here for completeness.

Theorem G.1 (Restatement of Theorem A.4 in [CC16]) *There is an efficient algorithm, that, given a graph G with maximum vertex degree at most d, an integer $q \geq 1$, and a set $\mathcal{P}$ of at least $16dq$ disjoint paths in G, computes a subset $\mathcal{P}' \subseteq \mathcal{P}$ of at least $|\mathcal{P}|/2$ paths, and a collection $\mathcal{C}$ of disjoint connected subgraphs of G, such that each path $P \in \mathcal{P}'$ is completely contained in some subgraph $C \in \mathcal{C}$, and each such subgraph contains at least q and at most $4dq$ paths in $\mathcal{P}$.*

Proof: Starting from G, we construct a new graph H, by contracting every path $P \in \mathcal{P}$ into a super-node u_P. Let $U = \{u_P \mid P \in \mathcal{P}\}$ be the resulting set of super-nodes. Let τ be any spanning tree of H, rooted at an arbitrary vertex r. Given a vertex $v \in V(\tau)$, let τ_v be the sub-tree of τ rooted at v. Let $J'_v \subseteq V(G)$ be the set of all vertices of τ_v that belong to the original graph G (that is, they are not super-nodes), and let J''_v be the set of all vertices of G that lie on paths $P \in \mathcal{P}$ with $u_P \in \tau_v$. We then let $J_v = J'_v \cup J''_v$. We also denote $G_v = G[J_v]$; observe that it must be a connected graph. Over the course of the algorithm, we will delete some vertices from τ. The notation τ_v and G_v is always computed with respect to the most current tree τ. We start with $\mathcal{C} = \emptyset, \mathcal{P}' = \emptyset$, and then iterate.

Each iteration is performed as follows. If $q \leq |V(\tau) \cap U| \leq 4dq$, then we add the graph G_r corresponding to the root r of τ to $\mathcal{C}$, and terminate the algorithm. If $|V(\tau) \cap U| < q$, then we also terminate the algorithm (we will show later that $|\mathcal{P}'| \geq |\mathcal{P}/2|$ at this point). Otherwise, let v be the lowest vertex of τ with $|\tau_v \cap U| \geq q$. If $v \notin U$, then, since the degree of every vertex in G is at most d, $|\tau_v \cap U| \leq dq$. We add G_v to $\mathcal{C}$, and all paths in $\{P \mid u_P \in \tau_v\}$ to $\mathcal{P}'$. We then delete all vertices of τ_v from τ, and continue to the next iteration.

Assume now that $v = u_P$ for some path $P \in \mathcal{P}$. If $|\tau_v \cap U| \leq 4dq$, then we add G_v to $\mathcal{C}$, and all paths in $\{P' \mid u_{P'} \in \tau_v\}$ to $\mathcal{P}'$ and continue to the next iteration. So we assume that $|\tau_v \cap U| > 4dq$.

Let $v_1, \ldots, v_z$ be the children of v in τ. Build a new tree τ' as follows. Start with the path P, and add the vertices $v_1, \ldots, v_z$ to τ'. For each $1 \leq i \leq z$, let $(x_i, y_i) \in E(G)$ be any edge connecting some vertex $x_i \in V(P)$ to some vertex $y_i \in V(G_{v_i})$; such an edge must exist from the definition of G_{v_i} and τ. Add the edge (v_i, x_i) to τ'. Therefore, τ' is the union of the path P, and a number of disjoint stars whose centers lie on the path P, and whose leaves are the vertices $v_1, \ldots, v_z$. The degree of every vertex of P is at most d. We define the *weight* of the vertex v_i as the number of the paths in $\mathcal{P}$ contained in G_{v_i} (equivalently, it is $|U \cap \tau_{v_i}|$). Recall that the weight of each vertex v_i is at most q, by the choice of v. For each vertex $x \in P$, the weight of x is the total weight of its children in τ'. Recall that the the total weight of all vertices of P is at least $4dq$, and the weight of every vertex is at most dq. We partition P into a number of disjoint segments $\Sigma = (\sigma_1, \ldots, \sigma_\ell)$ of weight at least q and at most $4dq$ each, as follows. Start with $\Sigma = \emptyset$, and then iterate. If the total weight of the vertices of P is at most $4dq$, we build a single segment, containing the whole path. Otherwise, find the shortest segment σ starting from the first vertex of P, whose weight is at least q. Since the weight of every vertex is at most dq, the weight of σ is at most $2dq$. We then add σ to Σ, delete it from P and continue. Consider the final set Σ of segments. For each segment σ, we add a new graph C_σ to $\mathcal{C}$. Graph C_σ consists of the union of σ, the graphs G_{v_i} for each v_i that is connected to a vertex of σ with an edge in τ', and the corresponding edge (x_i, y_i). Clearly, C_σ is a connected subgraph of G, containing at least q and at most $4dq$ paths of $\mathcal{P}$. We add all those paths to $\mathcal{P}'$, delete all vertices of τ_v from τ, and continue to the next iteration. We note that path P itself is not added to $\mathcal{P}'$, but all paths P' with $u_{P'} \in V(\tau_v)$ are added to $\mathcal{P}'$.

At the end of this procedure, we obtain a collection $\mathcal{P}'$ of paths, and a collection $\mathcal{C}$ of disjoint connected subgraphs of G, such that each path $P \in \mathcal{P}'$ is contained in some $C \in \mathcal{C}$, and each $C \in \mathcal{C}$ contains at least q and at most $4dq$ paths from $\mathcal{P}'$. It now remains to show that $|\mathcal{P}'| \geq |\mathcal{P}|/2$. We discard at most q paths in the last iteration of the algorithm. Additionally, when $v = u_P$ is processed, if $|\tau_v \cap U| > 4dq$, then path P is also discarded, but at least $4dq$ paths are added to $\mathcal{P}'$. Therefore, overall, $|\mathcal{P}'| \geq |\mathcal{P}| - \frac{|\mathcal{P}|}{4dq+1} - q \geq |\mathcal{P}|/2$, since $|\mathcal{P}| \geq 16dq$. □

We now turn to prove Theorem 6.10. Recall that we are given an α-Expanding Path-of-Sets System $\Sigma = (\mathcal{S}, \mathcal{M}, A_1, B_3)$ of width w and length 3, where $0 < \alpha < 1$, and the corresponding graph G_Σ has maximum vertex degree at most d. Our goal is to compute subsets $\hat{A}_1 \subseteq A_1, \hat{B}_3 \subseteq B_3$ of $\Omega(\alpha^2 w/d^3)$ vertices each, such that $\hat{A}_1 \cup \hat{B}_3$ is well-linked in G_Σ. Notice that we can assume w.l.o.g. that $w \geq 256d^3/\alpha^2$, as otherwise it is sufficient that each set $\hat{A}_1, \hat{B}_3$ contains a single vertex, which is trivial to ensure.

We apply Claim 6.9 to graph S_1, together with the sets A_1, B_1 of vertices, to compute a set $\mathcal{P}_1$ of $\lceil \alpha w/d \rceil$ node-disjoint paths in S_1, connecting vertices of A_1 to vertices of B_1. We then set $q = \lfloor 16d/\alpha \rfloor$, and use Theorem G.1, to compute a subset $\mathcal{P}'_1 \subseteq \mathcal{P}_1$ of at least $|\mathcal{P}_1|/2 \geq \alpha w/(2d)$ paths, and a collection $\mathcal{C}$ of disjoint connected subgraphs of S_1, such that each path $P \in \mathcal{P}'_1$ is completely contained in some subgraph $C \in \mathcal{C}$, and each such subgraph contains at least q and at most $4dq$ paths of $\mathcal{P}'_1$. (Note that from our assumption that $w \geq 256d^3/\alpha^2$, $|\mathcal{P}_1| \geq 16dq$). Clearly, $|\mathcal{C}| \geq \frac{|\mathcal{P}'_1|}{4dq} \geq \frac{\alpha^2 w}{256d^3}$. We select one representative path $P \in \mathcal{P}'_1$ from each subgraph $C \in \mathcal{C}$, so that $P \subseteq C$, and we let $\mathcal{P}^*_1 \subseteq \mathcal{P}'_1$ be the resulting set of paths. We are now ready to define the set $\hat{A}_1 \subseteq A_1$ of vertices: set $\hat{A}_1$ contains, for every path $P \in \mathcal{P}^*_1$, the endpoint of P that lies in A_1. Note that $|\hat{A}_1| = |\mathcal{P}^*_1| = |\mathcal{C}| \geq \frac{\alpha^2 w}{256d^3}$. For convenience, for every vertex $a \in \hat{A}_1$, we denote by $P_a \in \mathcal{P}^*_1$ the unique path originating at a, and we denote by $C_a \in \mathcal{C}$ the unique subgraph of S_1 containing P_a.

We select a subset $\hat{B}_3 \subseteq B_3$ of at least $\frac{\alpha^2 w}{256d^3}$ vertices similarly, by running the same algorithm in S_3. The set of paths obtained as the outcome of Theorem G.1 is denoted by $\mathcal{P}'_3$, and the set of connected subgraphs of S_3 by $\mathcal{C}'$. We also denote by $\mathcal{P}^*_3 \subseteq \mathcal{P}'_3$ the set of representative paths that we select from each subgraph of $\mathcal{C}'$. For every vertex $b \in \hat{B}_3$, we denote by $P_b \in \mathcal{P}^*_b$ the unique path originating at b, and we denote by $C_b \in \mathcal{C}'$ the unique subgraph containing P_b.

It remains to show that $\hat{A}_1 \cup \hat{B}_3$ is well-linked in G_Σ. We show this using the same arguments as in [CC16]. Let $X, Y \subseteq \hat{A}_1 \cup \hat{B}_3$ be two equal-cardinality sets of vertices. We need to show that there is a set $\mathcal{Q}$ of $|X| = |Y|$ disjoint paths connecting them in G_Σ, such that the paths in $\mathcal{Q}$ are internally disjoint from $\hat{A}_1 \cup \hat{B}_3$. We define a new subgraph $H \subseteq G_\Sigma$ as follows: graph H is the union of the graph S_2 and the matchings $\mathcal{M}_1$ and $\mathcal{M}_2$; additionally, for every vertex $v \in X \cup Y$, we add the graph C_v to H. It is now enough to show that there is a set $\mathcal{Q}$ of $|X| = |Y|$ disjoint paths connecting X to Y in H; such paths are guaranteed to be internally disjoint from $\hat{A}_1 \cup \hat{B}_3$. From the integrality of flow, it is sufficient to show a flow F in H, where every vertex in X sends one flow unit, every vertex in Y receives one flow unit, and every vertex of H carries at most one flow unit. We now construct such a flow. This flow will be a concatenation of three flows, F_1, F_2, F_3.

We start by defining the flows F_1 and F_3. Consider some vertex $v \in X \cup Y$, and assume w.l.o.g. that $v \in \hat{A}_1$. We select an arbitrary subset $U_v \subseteq B_1$ of $q = \lfloor 16d/\alpha \rfloor$ vertices that serve as endpoints of paths $P \in \mathcal{P}'_1$ that are contained in C_v. Since C_v is a connected graph, vertex v can send $1/q$ flow units to every vertex in U_v simultaneously, inside the graph C_v, so that the flow on every vertex is at most 1. We denote the resulting flow by F^v.

We obtain the flow F_1 by taking the union of all flows F^v for $v \in X$, and we obtain the flow F_3 by taking the union of all flows F^v for $v \in Y$ (we reverse the direction of the flow F^v in the latter case).

Let $R_1 = \bigcup_{v \in X} U_v$, and let $R_2 = \bigcup_{v \in Y} U_v$. Note that $R_1 \cup R_2 \subseteq B_1 \cup A_3$. For every vertex $x \in R_1 \cup R_2$ that lies in B_1, we let x' be the vertex of A_2, that is connected to x by an edge of $\mathcal{M}_1$. Similarly, for every vertex $x \in R_1 \cup R_2$ that lies in A_3, we let x' be the vertex of B_2, that connects to x by an edge of $\mathcal{M}_2$. Let $R_1' = \{x' \mid x \in R_1\}$ and $R_2' = \{x' \mid x \in R_2\}$. Note that R_1', R_2' are disjoint sets of vertices in S_2. Since graph S_2 is an α-expander, there is a flow F_2' in S_2, where every vertex in R_1' sends one flow unit, every vertex in R_2' sends one flow unit, and every edge carries at most $1/\alpha$ flow units. Scaling this flow down by factor $q = \lfloor 16d/\alpha \rfloor$, we obtain a new flow F_2 in S_2, where every vertex of R_1' sends $1/q$ flow units, every vertex of R_2' receives $1/q$ flow units, and every vertex of S_2 carries at most one flow unit.

The final flow F is obtained by concatenating the flows F_1, F_2 and F_3, and sending $1/q$ flow units on every edge of $\mathcal{M}_1 \cup \mathcal{M}_2$ that is incident to a vertex of $R_1 \cup R_2$. The flow in F guarantees that every vertex of X sends one flow unit, every vertex in Y receives one flow unit, and every vertex of G_Σ carries at most one flow unit.

H Proof of Observation 7.1

We start with the cut (X, Y) and perform a number of iterations. In every iteration, we modify the cut (X, Y) so that the number of connected components in $G \setminus E(X, Y)$ strictly decreases, while ensuring that the cut sparsity does not increase. We now describe the execution of an iteration. Let (X, Y) be the current cut. Let $\mathcal{C}_X$ and $\mathcal{C}_Y$ be the sets of all connected components of $G[X]$ and $G[Y]$ respectively. If $|\mathcal{C}_X| = |\mathcal{C}_Y| = 1$, then we return the cut (X, Y), and terminate the algorithm. We assume from now on that this is not the case.

Assume w.l.o.g. that $|X| \leq |Y|$. Let $\rho_X := \frac{|E(X,Y)|}{|X|}$ and $\rho_Y := \frac{|E(X,Y)|}{|Y|}$. We consider the following two cases.

Case 1: The first case happens when $|\mathcal{C}_X| > 1$. Recall that $|E(X, Y)| = \rho_X |X|$. Thus, there is a connected component $C \in \mathcal{C}_X$ such that $|E(C, Y)| \geq \rho_X |C|$. Consider a new partition (X', Y'), obtained by setting $X' = X \backslash C$ and $Y' = Y \cup C$. Notice that the number of connected components in $G \setminus E(X', Y')$ decreases by at least one. The sparsity of the new cut is:

$$\frac{|E(X', Y')|}{\min\{|X'|, |Y'|\}} = \frac{|E(X', Y')|}{|X'|} = \frac{|E(X, Y)| - |E(C, Y)|}{|X| - |C|} \leq \frac{\rho_X |X| - \rho_X |C|}{|X| - |C|} = \rho_X.$$

Case 2: If Case 1 does not happen, then $|\mathcal{C}_Y| > 1$ must hold. As before, there is a connected component $C \in \mathcal{C}_Y$ such that $|E(C, Y)| \geq \rho_Y |C|$. Consider the new partition (X', Y') by setting $X' = X \cup C$ and $Y' = Y \backslash C$. Notice that the number of connected components in $G \setminus E(X', Y')$ decreases by at least one. In order to bound the sparsity of the new cut, we consider two cases. If $|X'| \geq |Y'|$, then the sparsity of the new cut is

$$\frac{|E(X', Y')|}{|Y'|} = \frac{|E(X, Y)| - |E(C, Y)|}{|Y| - |C|} \leq \frac{\rho_Y |Y| - \rho_Y |C|}{|Y| - |C|} = \rho_Y \leq \rho_X.$$

Otherwise, the sparsity of the new cut is

$$\frac{|E(X', Y')|}{|X'|} = \frac{|E(X, Y)| - |E(C, Y)|}{|X| + |C|} < \frac{|E(X, Y)|}{|X|} = \rho_X.$$

It is immediate to verify that the algorithm is efficient, and that it produces the cut (X^*, Y^*) with the required properties.

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